Fibonacci and the Abacus Schools in Italy. Mathematical Conceptual Streams – Education and its Changing Relationship with Society – Almagest

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Abstract

In this paper we present the relations between mathematics and mathematics education in Italy between the 12th and the 16th century. Since the subject is extremely wide, we will focus on two case-studies to point out some relevant aspects of this phenomenon: 1) Fibonacci's studies (12th-13th century); 2) *Abacus schools*. More particularly, Fibonacci, probably the greatest European mathematician of the Middle Ages, made the calculations with Hindu-Arabic digits widely spread in Europe; *Abacus schools* were also based on the teaching of the calculation with Hindu-Arabic digits. These case-studies are significant for understanding the connections between science, science education and the development of science within Western civilization. We think that the knowledge of such significant relations can be useful for the scholars who are nowadays engaged in mathematics education and in the research field of science-society relations. Finally, we attempt to outline the interaction between mathematics education and advanced mathematics in that period, focusing on the figure of Leonardo Pisano (c. 1170-c.1250), called Fibonacci, who played an influential role both in mathematics education and in advanced mathematics.

An outline

The success of the work in which Fibonacci explained the Hindu-Arabic digits (*Liber Abaci*, 1202-1228)[1] and of the *Abacus schools* depended in great part on the particular economic Italian situation. In the period between the 13th and the 16th centuries Italy was an advanced country from an economic and scientific point of view. Hence, an examination of the links connecting mathematics, mathematics education, and social transformations during this epoch is important in order to capture certain aspects of the science-society relation.[2] The general phenomenon is complex and, probably, a general and univocal paradigm of explanation does not exist. Rather, different models and paradigms do exist. Through the history of mathematics and science, we try to point out one of these paradigms considering this particular period. Further researches will clarify if it can be extended to other periods, too.[3]

Education & science in context

Mathematics education is a social phenomenon because it is influenced by the needs of the labour market and by the basic knowledge of mathematics necessary for every person in order to be able to perform some operations which are indispensable in the social and economic daily life. Therefore, the way in which mathematics education is framed changes according to the modifications of the social environment and know-how. This statement on mathematics education is valid in our time, and applies also to the epoch and the place we are dealing with. Because, during the long period between the second half of the 12th century and the first half of the 16th century, Italy (or rather Italian cities-states and, afterwards, Italian regional states) was one of the most advanced countries in terms of economic structure, development of mathematics and science. An organized mathematics education was developed in some Italian regions, starting from the 13th century. This kind of education was connected to the economic and social structure. The way in which mathematics education was structured in Italy between the 13th and the beginning of the 16th century is significant and paradigmatic in order to highlight the influence society can have on education. Mathematics education was organized around the so called *scuole*

1 The first version of *Liber Abaci* dated back to 1202. This version is lost and only the second version, rewritten by Leonardo Pisano in 1228, is available (Fibonacci [1228] 1857). The first edition of the *Liber Abaci* was given by Boncompagni (Fibonacci [1228] 1857). An English translation of *Liber Abaci* is available (Siegler 2002).

2 For example, as regards to methods used in architecture during this period, and also in connection to science education and science-society relations (on that see Knobloch 1994).

3 This paper is a theoretical deepening and advancement of the issues that we have dealt with in two previous works Pisano and Bussotti 2014a,b, 2013a; see also Pisano and Capecchi 2015.

d'abaco (*Abacus schools*). Furthermore, the *Abacus schools* and the abacus treatises linked to them were also important for mathematical research. As a matter of fact, the authors of the abacus treatises' were, in most cases, teachers in the *Abacus schools*.

The heritage of the *Abacus schools* was influential for the mathematical education and the appearance of a number of important mathematicians who lived in the late middle Ages and in the Renaissance period. An emblematic case is that of Luca Pacioli (1445-1517) who, in his turn, played a fundamental role in Leonardo's da Vinci (1452-1519) (Pisano 2013) mathematical education. Furthermore, the history of the *Abacus schools* has connections (even though, in this case, they are more indirect than in Pacioli's case) with mathematicians such as Scipione dal Ferro (1465-1526), Niccolò Fontana called Tartaglia (1499-1557) (Pisano and Capecchi 2015), Gerolamo Cardano (1501-1576), Lodovico Ferrari (1522-1565), Rafael Bombelli (1526-1572), who developed algebra – and in particular who studied the solutions for equations of third and fourth degree. The relations between those mathematicians are significant from a scientific, social and anthropologic point of view. We will deal with them in two upcoming publications. Thus, the *Abacus schools* can be interpreted as the epicentres of complex phenomena related to mathematic, mathematics education and civilization in Italy from the late middle Ages to the early modern age.

Fibonacci: The Connections between Theoretical Mathematics and Mathematics Education

Leonardo Pisano, known as Fibonacci, is one of the protagonists of the story we are narrating because his contributions to the spread of the Hindu-Arabic numerals and to the calculations on the lines with the decimal positional system was of fundamental importance. We will analyze the technique used by Fibonacci to carry out certain operations. In addition, Fibonacci contributed decisively to advanced mathematics. These contributions were, at least in part, tied to his contributions to mathematics education. An interesting link between mathematics education and advanced mathematics was thus forged by Leonardo. Actually, in order to appreciate fully Fibonacci's results, it is necessary to explain how operations with the *Abacus* were carried out and to examine –briefly– how the technique of calculation with an *abacus* was improved around the 11th century. This will be useful for the readers, since it will enable them to fully capture the similarities and the differences between an advanced calculation with an *abacus* and the calculation on the lines introduced in Europe by Fibonacci.[4]

4 Leonardo was not the only responsible for the introduction of the decimal positional system and the operations in the lines. The phenomenon is complex and connected to socio-economical aspects. Furthermore, with regard to this particular context, historical evidence is largely lacking for the 12th and 13th centuries. The important works by Jens

A Protohistory: calculating with an Abacus

Until the 13th century, the decimal positional system and the Indo Hindu-Arabic numerals were not widely spread in Europe. From the 11th to the 13th century the four arithmetical operations were carried out using the *Abacus*. This instrument provides the possibility to develop all the four operations through standard procedures and its basis is a positional idea of the number because every column or line (it depends on the kind of *Abacus*) can represent a power of the number assumed as the basis of the system.[5] However the possibility to operate on the lines does not exist with the *Abacus* and the step from the representation of a number on an *Abacus* in a positional form to the possibility to write a number in a positional form is not trivial at all, because this step implies the full comprehension of the role of the digit zero inside the positional system. This is not necessary for calculations with the *Abacus*, but it is if wants to write the positional form of a number. The multiplication and, moreover, the division become complicated with the *Abacus*. The division can be carried out only by successive subtractions and this implies, in general, a long series of operations. The situation is even more complicated for other operations, such as the calculation of the square roots. Furthermore, if one has to represent a number with many decimal positions or to carry out an operation for which many decimal positions are necessary, there is also a practical problem: it is needed to use more than one *Abacus* at the same time. The history of the calculations with the *Abacus* is an interesting chapter in the history of mathematics and the results one can obtain with this instrument are amazing, but they are subject to limitations which do not exist for the calculation in the lines. The *Abacus*, in various forms, was used in the Greek and Roman society, but it disappeared almost entirely in the early medieval Europe. It was reintroduced by Gerbert d'Aurillac (955-1003) –the Pope Sylvester II– who had spent a period of his life in the Catalan abbeys where he came into contact with the Arabic civilization[6] and

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Høyrup propose a different interpretation of the relations between Fibonacci and the abacus treatises (Høyrup 2003, 2004, 2007, 2009, 2010). Retrieved via: http://akira.ruc.dk/~jensh/Selected%20themes/Abbacus%20mathematics/index.htm. While referring to Høyrup's work, the date used indicates the year in which the work was originally written and the form in which it can be consulted in the mentioned site. In the references we provide bibliographical information on the published versions of these papers. We will deal with Høyrup's works in the next section.

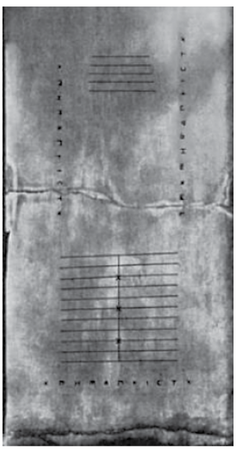
5 Among the numerous books which explain the calculation with the abacus, a text with a high pedagogical value is that by Schärlin (Schärlin 2003).

6 The present paper on Fibonacci and the abacus schools touches upon questions addressed by a wide-ranging series of studies, conducted either by other scholars or based on our own research, also involving the relations between Arabic and European science, in particular as far as the transformation of mechanics from the Arabic tradition to the Renaissance until the 17th century is concerned. See: Abattoy 2006, Abattoy, Renn and Weining 2001, Bussotti and Pisano 2013ab, 2014b, Capecchi and Pisano 2010, Elazar 2008, Pacioli 2004, Pisano 2013, 2014, Pisano and Bussotti 2012, 2015, Pisano and Capecchi 2010, Pisano and Gaudiello 2009, Van Egmont 2001.

where he learnt the decimal positional system. It is still uncertain whether he fully understood the role of zero, but certainly he realized the importance of the calculus with *Abacus*. He invented a particular kind of *Abacus* that was spread in Europe from the 11th to the 13th century. However, the spread of Gerbert's and his disciples' works was likely rather limited.[7]

To fully appreciate the conceptual and practical differences between the calculation with an abacus and the calculation on the lines within a positional system, we will try to explain how calculations with an abacus are performed. We will refer to Salamina *Abacus* and to Gerbert's *Abacus*: the former is an ancient Greek abacus, while the latter represents an improvement of the classical abacuses and a step towards the calculation with a positional system.

The Salamina *Abacus* was discovered in the island of Salamis (Greek) in 1846. It is presently held by the National Epigraphic Museum in Athens.



This *Abacus* is a rectangular tablet:
altitude 149 cm.; basis 75 cm.; thickness 4.5 cm.
Five parallel lines are drawn in the higher part;
at 50 cm from the last of these lines, a series
of other eleven parallel lines are drawn.
They are bisected by a perpendicular line.
The third, the sixth and the ninth of
these eleven lines are marked by crosses.
Three series of Greek characters are legible on the left,
on the right side and in the final part of *Abacus*' basis.

The most complete series has 13 characters:

These symbols represent numerical signs by means of which it is possible to calculate with the Greek money subdivisions: talents, drachmas, oboluses; khalkos.

1 talent = 6000 drachmas
1 drachma = 6 oboluses
1 obolus = 8 khalkos.

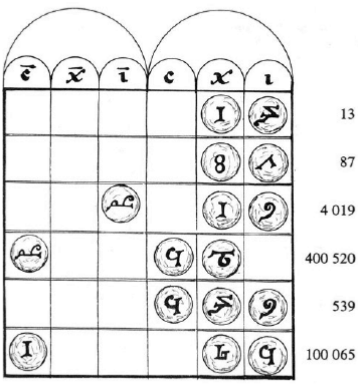
The value of the numerical symbols is expressed by the table.

Figure 1: *Salamis Tablet*[8]

7 As to Gerbert, see Pekonen 2000, Atkinson 2005, Vallicrosa 1931, Baird, 1994, Seife 2000, Flusche 2006, Brown, 2010. As to Gerbert Abacus in particular, see Folkerts 1996, Burnett 2002, Bramanti 2008, Brown 2010, 100-118, Nuno Silva 2013, Clagett.

8 Retrieved via: http://www.ee.ryerson.ca:8080/~elf/abacus/history.html. Retrieval date: 13/10/2015. The oldest counting board is made of marble (The National Museum of

Once introduced, this complex mechanism rendered possible the development of money calculations following the ordinary principles relative to the calculus with the use of an *Abacus*. Therefore, the tokens representing the numbers had a value according to their position in the columns. The idea of a numerical basis for the system is not completely developed because in some cases the basis is 10 (talents and drachmas), in other cases (the submultiples of the drachmas) the basis is 6 and its multiples, not its powers.
In the following section, we will present the *Abacus* envisaged by Gerbert which can be considered an intermediate phase between the calculation with an ordinary *Abacus* and the calculation on the lines.



This *Abacus* is composed by a board.
In the semicircles over the board the successive powers of ten are represented. Thus

$$I = 10^0;\ X = 10^1;\ C = 10^2$$

$$\overline{I} = 10^3;\ \overline{X} = 10^4;\ \overline{C} = 10^5$$

This is not a novelty for an *Abacus*. However, the tokens that represent the numbers are not equal, as it is the case with an ordinary *Abacus*, but on each of them a digit representation is inscribed.
The representation of zero is not inscribed. In its place there is a square without any token: in the referred figure we have, respectively in the lines, the numbers 13, 87, 4019, 400520, 539, 100065 (written in Indian digits of the tenth century).

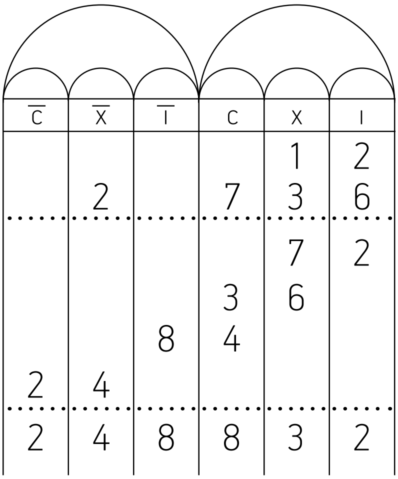
Figure 2: *Gerbert's abacus*[9]

In order to explain how Gerbert *Abacus* works, let us refer to the following figure in which, for convenience of the reader, the numbers are rendered in modern Arabic numerals.

Let us take the following example. We have to multiply 12 by 20726. The numbers are written in Gerbert *Abacus* as indicated above.

Epigraphy, Athens, Greece). Recently: http://www.dti.unimi.it/~citrini/Tesi/r9/index.html. Retrieval date: 13/10/2015.

9 Nuno Silva 2013, 112; see also Bramanti 2008.



First of all Gerbert writes the number of unities:

6x2 = 12

Then he writes 2 under the broken line in the column I and carries 1. The tens are given by

6x1+1 carried[10]

This is the 6 written under the 7 in the same column. Thus,

7+6 = 13, let us write 3 and carry 1.

With regard to the hundreds, they are given by 3x1+7x2+1. The 3 written under the broken line in the column C is relative to the product 3x1, the 4 under the 3 by the product 7x2 (4 and carry 1) the 8 under the second broken line is given by

3+4+1 (carried by the multiplication of the tens).

Here the carried number is written directly, being added to the addition 3+4, while, in the case of the tens, it had been added to the product 6x1. Of course, both theseways of writing are correct. By proceeding in this manner, one getsthe numbers of thousands, ten–thousands and hundred–thousands.

Figure 3: *Presentation of Gerbert abacus*[11]

Gerbert did not write the numbers as we do, but he still used tokens on which a numerical value was inscribed. However, Gerbert's *Abacus* is an important step in the history of the calculation by means of the decimal positional system. The literature (see ft. 7) on the way in which Gerbert knew and used the decimal positional system is abundant and the problems associated with Gerbert as a Pope and mathematician are interesting for the history of science and civilization. Before the reintroduction of the *Abacus*, the calculations in Europe during the middle ages were developed with the use of Roman numerals and, although a certain amount of studies has shown that many operations are possible with the Roman numerals,[12] these calculations are far more limited than those permitted by the *Abacus*.

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10 Pay attention that this is the 7 written under the broken line in the column X plus 3x2 = 6, which are tens, too

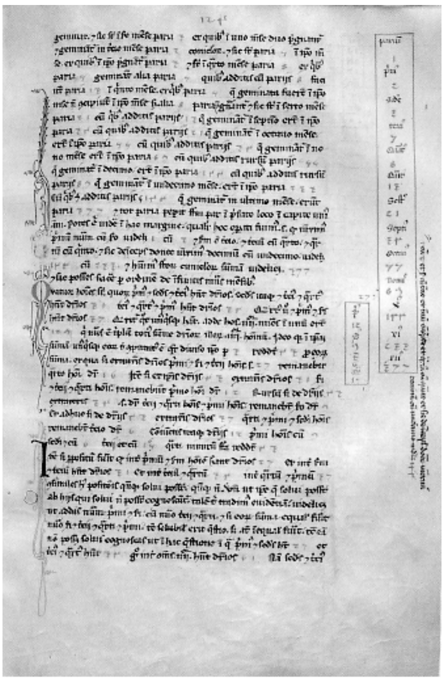
11 Nuno Silva 2013, 113..

12 A certain amount of material can be found on the Internet as for the calculation with the use of Roman numerals. There are also websites in which a *Roman calculator machine* can be found. With regard to the calculation with Roman numerals, without resorting to

Fibonacci: the introduction of the decimal positional system and the operations on the lines

The spread in Europe of the Hindu-Arabic numerals and positional system is a clear example indicating how great historical events are often due to a plurality of circumstances, in our case:

1. The cleverness of Fibonacci;
2. The events of his life;
3. The economic and social situation of the Italian maritime republics and the specific situation of Toscana in the 12th-13th centuries.



Leonardo Pisano fully realized the theoretical and practical importance of the calculus on the lines.

He understood the role of zero and the great progress that the numerical Hindu-Arabic system would have brought to mathematics and –more generally– to civilization.

The book in which the Arabic numerals and all the operations were presented is Fibonacci's *Liber Abaci*.

Figure 4: *The Liber Abaci*[13]

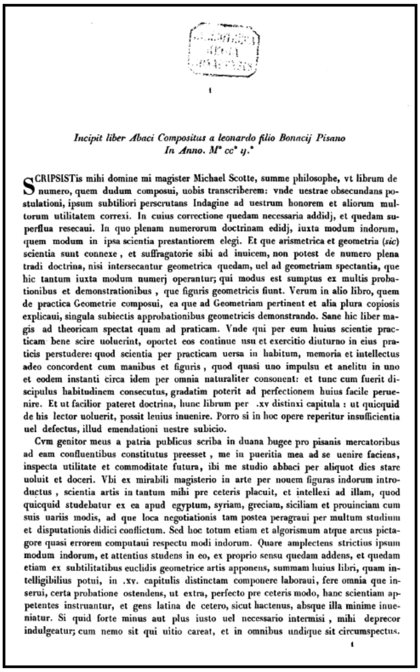
In the 12th century Pisa was a maritime republic and an important trade centre, having many

an Abacus, we refer to the lucid paper written by Turner (2007). It is necessary to add that Turner himself considers the operations with Roman numerals a curiosity rather than a historical reality, because, although the four arithmetical operations seem to be possible, they are far more complicated than the operations with the use of *Abacus*.

13 The Manuscript is preserved in the National Library, Firenze. *Codice Magliabechiano,* cs cl 2616, 124r.

relations with the Muslim world. The father of Leonardo, Guglielmo Bonacci or Bonaccio –from which the name Fibonacci as *Filius Bonacci*, which is to say, belonging to the Bonacci family or to the lineage of Bonaccio, likely derived[14]– was in the 80s of the 12th century *publicus scriba* (namely a delegate of Pisa Republic) in the important business city of Bugia, Algeria. He was responsible for the trade of Pisa in that area. At that time Leonardo lived with his father in Bugia and his interest in, and knowledge of, the Hindu-Arabic system dates back to this period. In the *Incipit* of the *Liber abaci* Leonardo wrote

When my father was given the task of *publicus scriba* in the Custom-office of Bugia for the Pisan merchants who arrived at that city, he made me to reach him when I was a child. Here he wanted me to dedicate a period to the study of the Abacus and to be instructed in this subject. There I was introduced to the nine figures of the Indians by an admirable teaching in that theory. I liked the knowledge of that subject more than all others and I was so engaged to understand that I learnt everything which had been studied about it in its different forms, in Egypt, Syria, Greece, Sicily and Provence, with great application and through the method of the debate. Later on I travelled for long periods to those trade–places. (Fibonacci [1228] 1857, 1)



Incipit liber Abaci Compositus a leonardo filio Bonacij Pisano
In Anno. M° cc° ij.°

SCRIPSISTis mihi domine mi magister Michael Scotte, summe philosophe, vt librum de numero, quem dudum composui, uobis transcriberem: vnde uestrae obsecundans postulationi, ipsum subtiliori perscrutans Indagine ad uestrum honorem et aliorum multorum utilitatem correxi. In cuius correctione quedam necessaria addidj, et quedam superflua resecaui. In quo plenam numerorum doctrinam edidj, iuxta modum indorum, quem modum in ipsa scientia prestantiorem elegi. Et que arismetrica et geometria (*sic*) scientia sunt connexe, et suffragatorie sibi ad inuicem, non potest de numero plena tradi doctrina, nisi intersecantur geometrica quedam, uel ad geometriam spectantia, que hic tantum iuxta modum numerj operantur; qui modus est sumptus ex multis probationibus et demonstrationibus, que figuris geometricis fiunt. Verum in alio libro, quem de practica Geometrie composui, ea que ad Geometriam pertinent et alia plura copiosis explicaui, singula subiectis approbationibus geometricis demonstrando. Sane hic liber magis ad theoricam spectat quam ad praticam. Vnde qui per eum huius scientie practicam bene scire uoluerint, oportet eos continue usu et exercitio diuturno in eius praticis perstudere: quod scientia per practicam uersa in habitum, memoria et intellectus adeo concordent cum manibus et figuris, quod quasi uno impulsu et anelitu in uno et eodem instanti circa idem per omnia naturaliter consonent: et tunc cum fuerit discipulus habitudinem consecutus, gradatim poterit ad perfectionem huius facile peruenire. Et ut facilior pateret doctrina, hunc librum per .xv distinxi capitula: ut quicquid de his lector uoluerit, possit leuius inuenire. Porro si in hoc opere reperitur insufficientia uel defectus, illud emendationi uestre subicio.

Cvm genitor meus a patria publicus scriba in duana bugee pro pisanis mercatoribus ad eam confluentibus constitutus preesset, me in pueritia mea ad se uenire faciens, inspecta utilitate et commoditate futura, ibi me studio abbaci per aliquot dies stare uoluit et doceri. Vbi ex mirabili magisterio in arte per nouem figuras indorum introductus, scientia artis in tantum mihi pre ceteris placuit, et intellexi ad illam, quod quicquid studebatur ex ea apud egyptum, syriam, greciam, siciliam et prouinciam cum suis uariis modis, ad que loca negotiationis tam postea peragraui per multum studium et disputationis didici conflictum. Sed hoc totum etiam et algorismum atque arcus pictagore quasi errorem computaui respectu modi indorum. Quare amplectens strictius ipsum modum indorum, et attentius studens in eo, ex proprio sensu quedam addens, et quedam etiam ex subtilitatibus euclidis geometrice artis apponens, summam huius libri, quam intelligibilius potui, in .xv. capitulis distinctam componere laboraui, fere omnia que inserui, certa probatione ostendens, ut extra, perfecto pre ceteris modo, hanc scientiam appetentes instruantur, et gens latina de cetero, sicut hactenus, absque illa minime inueniatur. Si quid forte minus aut plus iusto uel necessario intermisi, mihi deprecor indulgeatur; cum nemo sit qui uitio careat, et in omnibus undique sit circumspectus.

Figure 5: *The first page of* the *Liber Abaci*[15]

This quotation is suggestive. Leonardo was impressed by the advantages deriving from the "nine figures of the Indians". Therefore he learnt all the secrets of this decimal positional system and decided to write a book on this subject with *Euclidean rigour*, so as to make the positional system known in Europe. This book was –as we will see– a masterpiece in mathematics education because all the fundamental properties and operations with

14 As to the names of Leonardo, see Boncompagni 1852. The first 23 pages are dedicated to this subject. See also Franci 2002a, Giusti 2002, Ambrosetti 2008, 218-219.

15 For the sake of brevity, we have provided the English translation of the most important part, the second paragraph – starting from "Cum genitor meus ..." – of Fibonacci's page, since the first one is not important in our context.

numbers are explained, but at the same time it opened a new perspective for advanced mathematics due to the numerous difficult problems posed and to Leonardo's rigorous way of thinking.

The *Liber Abaci* is only the beginning of Leonardo's *mathematical career*. For, he wrote several works: the *Practica Geometriae* in 1220-1221 (Fibonacci 1862), a book (as the title itself suggests) dedicated to practical geometry. Fibonacci teaches how to calculate the surfaces of the most common plane figures and how to calculate the volumes of some common solids. Although the treatment concerns elementary properties of the figures, it is interesting because of its completeness, rigour and didactical inspiration.

Despite the fact that the *Practica Geometriae* is not a revolutionary book, as the *Liber Abaci* was, it has to be considered as an important step in the history of mathematics education. In 1226 the famous meeting between the Emperor Friedrich the second and Fibonacci took place in Pisa. This meeting was not important only for Leonardo's life, but for the history of mathematics, too, because Fibonacci had close relations with important mathematicians and scientists working at Friedrich's court. The mathematician Giovanni da Palermo challenged Leonardo to solve three problems, the solution of which offered Fibonacci the occasion to write his two most profound books, from a mathematical point of view: the *Flos* (Fibonacci 1862) and the *Liber Quadratorum* (Fibonacci 1862). The *Liber Quadratorum* was written in 1225, as Leonardo explicitly writes in the *incipit*. The *Flos* can be reasonably dated to the same period. The problems posed were (in modern language) the following:

1. To solve the following equation:

$$x^3+2x^2+10x=20\cdot$$

2. To solve the following series of equations:

$$x_1+\frac{1}{2}(x_2+x_3+x_4)=x_2+\frac{1}{3}(x_3+x_4+x_5)=x_3+\frac{1}{4}(x_4+x_5+x_1)=$$
$$x_4+\frac{1}{5}(x_5+x_1+x_2)=x_5+\frac{1}{6}(x_1+x_2+x_3);$$

3. To solve- in integers or rational numbers - the following two equations:

$$x^2+5=y^2;$$
$$x^2-5=z^2;$$

The problems 1 and 2 were solved in the *Flos*. The third one, together with some other difficult questions of Diophantine analysis, in the *Liber Quadratorum*. The problems belonged to the advanced mathematics research field and we will not deal with them. Here it suffices to point out that the profoundness and the results obtained by Leonardo were overcome only by the great French mathematicians of the 17th century and, in particular, as for Diophantine analysis, by Fermat.[16]

In 1228 Fibonacci wrote a second version of the *Liber Abaci* for Michele Scotus, one of the most important scientists (he was an astronomer and astrologer) in the court of Friedrich. Leonardo also wrote a letter to Maestro Teodoro d'Antiochia (c. 1155/1158-1246) a court mathematician of the Emperor Friedrich. Here he addressed a series of problems similar to those dealt with in the second part of the *Flos*. The most interesting consists in inscribing an equilateral pentagon in a given isosceles triangle. Fibonacci also composed two works which are now lost:

α) *Libro di merchaanti o detto di minor guisa*;
β) *Libro sopra il 10mo di Euclide*.

In 1241 Leonardo was still alive, as it is shown by the fact that the municipality of Pisa decided to renew the annual salary for Leonardo Pisano Bigollo[17] because of his important scientific merits, which had procured a great prestige to Pisa and because of his significant work as a business adviser to the Municipality. Leonardo died probably between 1245 and 1250 (Franci 2002a). After the *Liber Abaci*, from the 13th to the 16th century, a series of treatises were written by many mathematicians where the use of the Hindu-Arabic numerals and the operations on the lines were expounded. All these books treated these subjects in a less profound and rigorous manner than the *Liber Abaci*. However, Fibonacci's work was an explicit reference point for many of them. The advantages obtained by the calculations on the lines were appreciated only after Leonardo's results were popularized and written in vulgar Italian. Because the abacus treatises are written in Italian. The question is: *why was there an interest in a new and quick way of calculating?*

The 12th and the 13th centuries were, in Italy (and in particular in the Northern and Central Italy), the epoch of the cities-states**.** Commerce was a fundamental part of economic life of the Italian cities-states. This was the case with every Italian city-state, and especially for the four maritime republics (Amalfi, Genova, Pisa, Venezia). Until the defeat at Meloria against Genova in 1284, Pisa had a key role in the trade with the Western Muslim world. This was the reason why Leonardo's father worked in Bugia. When the economy of a certain region is based on trade, the ability to calculate in an easy, quick

16 Concerning the contributions of Leonardo to advanced mathematics, see Woepcke 1853, Woepcke 1854-1855, Genocchi, 1855abc, Genocchi 1857, Woepcke 1860-61, Ver Eecke in Fibonacci 1952, Favaro 1874, Picutti 1979, 1983, Lüneburg 1991, Rashed 1994, 2003, Bussotti 2003, 2004, 2008, 2009. As for Fermat's number theory and his relations with the mathematical knowledge preceding his works, see the first part of the chapter *Fermat* by Bussotti (Bussotti 2006).

17 Bigollo was another name by which Leonardo was known (Boncompagni 1852, 1-25).

and precise manner becomes indispensable. Therefore, as already mentioned above, the introduction of the decimal positional system in Europe was due to:

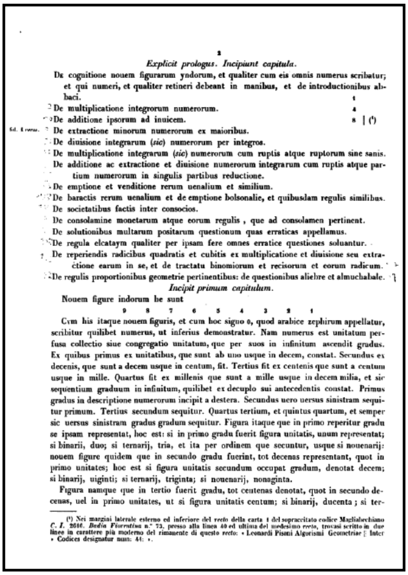
1) The opportunity Fibonacci had to visit Arabic countries where this system had been imported from India.[18]
2) His cleverness in understanding the conspicuous advantages deriving from the calculations on the lines with the use of a positional system.
3) To the socio-economic situation of a whole Italian region in which commerce was the main economic activity.

On the other hand, the success of the re-introduction of the *abacus* in Europe by Gerbert d'Aurillac in the 11th century is also connected with the changes that were transforming the European civilization. Because we know that around the 11th century the European population had a consistent increment, many cities in the Flanders and in the North of the France became important textile manufacturing centers, the agricultural technique was improved and some Italian cities (the most important was Venezia) became world trading-centers. As a consequence of these facts, trading relations became more important and –above all– more extended (the *international* commerce begins to be an economic business) than in the Early Middle Ages. Commerce, the necessity to exchange goods and money, to convert the value of a money into another money, brought about, in an initial phase, the reintroduction of the *Abacus* and in a second phase, also thanks to Fibonacci, the introduction of the positional system and the operations on the lines.

18 This opportunity, in its turn, presented itself owing to Pisa's economic situation and to the profession of Leonardo's father.

The Content of the Liber Abaci

The *Liber Abaci* is divided into 15 chapters:



2

Explicit prologus. Incipiunt capitula.

De cognitione nouem figurarum yndorum, et qualiter cum eis omnis numerus scribatur; et qui numeri, et qualiter retineri debeant in manibus, et de introductionibus abbaci. 1

De multiplicatione integrorum numerorum. 4

De additione ipsorum ad inuicem. 8 | (¹)

De extractione minorum numerorum ex maioribus.

De diuisione integrarum (*sic*) numerorum per integros.

De multiplicatione integrarum (*sic*) numerorum cum ruptis atque ruptorum sine sanis.

De additione ac extractione et diuisione numerorum integrarum cum ruptis atque partium numerorum in singulis partibus reductione.

De emptione et venditione rerum uenalium et similium.

De baractis rerum uenalium et de emptione bolsonalie, et quibusdam regulis similibus.

De societatibus factis inter consocios.

De consolamine monetarum atque eorum regulis, que ad consolamen pertinent.

De solutionibus multarum positarum questionum quas erraticas appellamus.

De regula elcataym qualiter per ipsam fere omnes erratice questiones soluantur.

De reperiendis radicibus quadratis et cubitis ex multiplicatione et diuisione seu extractione earum in se, et de tractatu binomiorum et recisorum et eorum radicum.

De regulis proportionibus geometrie pertinentibus: de questionibus aliebre et almuchabale.

Incipit primum capitulum.

Nouem figure indorum he sunt

9 8 7 6 5 4 3 2 1

Cvm his itaque nouem figuris, et cum hoc signo 0, quod arabice zephirum appellatur, scribitur quilibet numerus, ut inferius demonstratur. Nam numerus est unitatum perfusa collectio siue congregatio unitatum, que per suos in infinitum ascendit gradus. Ex quibus primus ex unitatibus, que sunt ab uno usque in decem, constat. Secundus ex decenis, que sunt a decem usque in centum, fit. Tertius fit ex centenis que sunt a centum usque in mille. Quartus fit ex millenis que sunt a mille usque in decem milia, et sic sequentium graduum in infinitum, quilibet ex decuplo sui antecedentis constat. Primus gradus in descriptione numerorum incipit a destera. Secundus uero uersus sinistram sequitur primum. Tertius secundum sequitur. Quartus tertium, et quintus quartum, et semper sic uersus sinistram gradus gradum sequitur. Figura itaque que in primo reperitur gradu se ipsam representat, hoc est: si in primo gradu fuerit figura unitatis, unum representat; si binarii, duo; si ternarij, tria, et ita per ordinem que secuntur, usque si nouenarij: nouem figure quidem que in secundo gradu fuerint, tot decenas representant, quot in primo unitates; hoc est si figura unitatis secundum occupat gradum, denotat decem; si binarij, uiginti; si ternarij, triginta; si nouenarij, nonaginta.

Figura namque que in tertio fuerit gradu, tot centenas denotat, quot in secundo decenas, uel in primo unitates, ut si figura unitatis centum; si binarij, ducenta; si ter-

(¹) Nei margini laterale esterno ed inferiore del *recto* della carta 1 del sopraccitato codice Magliabechiano C. I. 2616. *Badia Fiorentina* n.° 73, presso alla linea 40 ed ultima del medesimo *recto*, trovasi scritto in due linee in carattere più moderno del rimanente di questo *recto*: « Leonardi Pisani Algorismi Geometriae [Inter « Codices designatur num: 44: ».

1. Introduction to the Indian figures and the way of writing them write them, introduction to the use of the *Abacus*;
2. Multiplication of integers;
3. Addition of integers;
4. Subtraction of integers from integers;
5. Division of integers by integers;
6. Multiplication of integer numbers by fractions and of fractions by fractions;
7. Addition, subtraction and division of integers by fractions and reduction of fractions to other fractions;
8. Purchase and sale of goods and similar questions;
9. Barter of goods and purchase of *bolsonalia* (money whose value was calculated according to the quantity of silver present in them; seeVianello 1896, 93) and their rules;
10. Partnerships between associates;
11. Coining of money and rules concerning coining;
12. Solutions of many different questions that we call *erraticas*[19];
13. Rule elcataym (double false position) and the manner in which almost all *erraticae questiones* can be solved by means of it;
14. Extraction of square and cubic roots. Treatise of the binomials;
15. Rules concerning the geometrical proportions: questions of algebra and almucabala.

Figure 6: *The Prologue of the Liber Abaci*[20]

19 The *erraticae questiones* are mathematical problems of various kinds, having some similarities, but not strictly connected the one with the other.

20 Fibonacci [1228] 1857, 2.

The *Liber Abaci* can be considered as an *advanced handbook*, whose aim is to educate the Western people –and above all the mathematicians and the merchants– about the calculus with Hindu-Arabic numerals. The chapters concerning partnerships, coining of money and exchange of money, assert that this book had the purpose to introduce the merchants to a set of knowledge that could be of fundamental importance to them. Nevertheless, some of the *erraticae questiones*, the part dedicated to geometry, the rigour with which the whole treatise is presented and the use itself of Latin, make the *Liber Abaci* a difficult text for non-expert mathematicians and learned men. This is why many of the treatises that were published after *Liber Abaci* were written in vernacular languages and were of a more practical and less rigorous and theoretical character than Leonardo's.

The voluminous size of the book, its language and Leonardo's way of reasoning, in which both Hindu-Arabic and classical Greek mathematics converge with original Fibonacci's ideas, make the *Liber Abaci* a book that cannot be easily interpreted and fit smoothly into the history of mathematics. We must also take into consideration that there is not yet a critical edition of this text. To give the reader an idea how Leonardo worked we will provide two examples drawn from the *Liber Abaci*:

1) Multiplication among integers.
2) Extraction of square root.

We are convinced that an analysis of the way in which Fibonacci carried out the operations is necessary in order to clearly explain his methods. More particularly, we have chosen two examples: the first concerns multiplication. In this case, Fibonacci's method became one of the standard methods in the mathematical teaching carried out in the abacus schools. The second example regards the extraction of the square root, a problem, which, at that time, was rather difficult. The manner in which Leonardo develops this operation offers, thus, the reader the possibility to appreciate his ingenuity. Furthermore, it clarifies our assertion that the *Liber Abaci* is a text where mathematics education and advanced mathematics are closely connected. The *Liber Abaci* offers plenty of interesting examples and the choice is, to some extent, arbitrary. We have chosen two operations which, in our opinion, provide a good starting point for getting a graspon the conceptual, operative and methodological aspects of Fibonacci's work.

An example of Multiplication between integers from the Liber Abaci

The chapter concerning multiplication is the second one. Leonardo claims that it is divided into eight parts, but the eighth part is missing and its content is included in the fifth one. In the first part, Fibonacci deals with the multiplications between two numbers with two digits, in the second part, with three digits, in the third, with four, in the fourth, with five, in the fifth, the method is extended to an arbitrary number of digits, whereas in the sixth and the seventh he shows how to multiply *cordetenus*.[21] Every part is divided into subparts: for example, in the first part, Fibonacci shows separately how to multiply two equal numbers of two digits, how to multiply two different numbers of two digits and how to multiply one number of one digit by one number of two digits. Every explanation is detailed and almost pedantic to our eyes. This indicates how much Leonardo was willing to introduce the reader into all the details of the properties of the decimal system and the operations performed in it. One example of multiplication between two numbers with two digits will clarify Fibonacci's method.

Leonardo proposed to multiply 37 by 49 (Fibonacci [1228] 1857, 9-10). He proceeds in the following way:

9

perius inuenimus: uel ex dicta summa prohice 9, remanebunt 136, de quibus deme 3 et 6 cum coniunta faciant 9, remanebit similiter 1 indiuisibiles de 1369 diuisis per 9.

Item si multiplicare uis 98 per 98, scribantur ut predixi 98 sub 98 et multiplicentur 8 per 8, erunt 64: ponantur 4 super utrumque 8, et seruentur pro decenis in manu 6, et multiplicentur 8 per 9, erunt 72; et iterum econtra multiplicentur 8 de inferiori per 9 de superiori, erunt similiter 72, que iungantur cum aliis 72 et cum 6 in manu seruatis, erunt 150; et cum non sit unitas in predictis 150, ponendum est zephyrum super utrumque 9, et seruentur pro decenis in manu 15, et multiplicentur 9 per 9, erunt 81, que addantur cum 15 in manu seruatis, erunt 96, de quibus 96 scribantur 6 in tertio gradu et 9 in quarto, ut in hac descriptione cernitur. Modo uideamus si hec multiplicatio recta est: iungantur figure de superiori 98, scilicet 9 cum 8, et dematur 9, remanebunt 8. Iterum illud idem fiat de inferioribus 98, remanebunt similiter 8; et multiplicentur 8 per 8, erunt 64, de quibus extrahantur omnes nouene que sunt in eisdem 64, remanebit pro pensa 1, uel aliter: iungantur figure que sunt in predictis 64, scilicet 6 cum 4, erunt 10, de quibus demantur 9, remanebit similiter 1, postea colligantur figure, que sunt in summa multiplicationis, scilicet 9 et 6 et 0 et 4 tamen non est necesse ut figura nouenarii colligatur in aliqua persimili probatione, cum nouenarius semper erit, ut extrahi precipiatur unde colligantur 6 et 0 et 4, erunt 10, de quibus demantur 9, remanebit 1 pro pensa, sicuti remanere oportebat. Cum autem uolueris multiplicare aliquem numerum de secundo gradu in se non habentem unitates, uel in primo gradu, ut in 10 et 40 uel 90, in quorum capitibus zephyrum semper esse necesse est, sic erit faciendum: scribarum (*sic*) numerum hoc ut supra dixi; et multiplicabitur secundus gradus per secundum tantum, et ponatur ante summe duo zephyra, et sic habebimus summam cuiuslibet dictarum multiplicationum. Vt pote si queratur multiplicatio de 70 in 70, scribantur itaque utraque 70 supradicto modo, et multiplicetur figura septenarii que est in secundo gradu superiori (*sic*) numeri per 7 inferioris, erunt 49, ante quem numerum ponantur duo zephyra, scilicet pro his que sunt ante utraque 7, faciunt 4900, que sunt summa quesite multiplicationis, scilicet queratur | de 37 in 49, scribantur 49 sub 37, scilicet maior numerus sub minori, et similis gradus sub simili gradu, ut in hac margine cernitur; et multiplicentur 7 per 9, erunt 63; ponantur 3 super 7, et pro decenis seruentur in manu 6, et multiplicentur 7 per 4 in cruce, erunt 28, que addantur cum 6 in manu seruatis, erunt 34. Item multiplicentur 9 per 3, erunt 27, que addantur cum 34, erunt 61: ponatur 1 super 3 et pro decenis seruentur in manu 6, et multiplicentur 3 per 4, erunt 12, que addat cum 6, erunt 18, que ponat post 13 superius posita, egredientur pro summa dicte multiplicationis 1813, ut hic ostenditur.

Et si multiplicatio recta est, ita cognoscitur: diuidantur 37 per 9, hoc est, addantur figure de 37, scilicet 3 cum 7, erunt 10, de quibus demantur 9, remanebit 1, quod seruet; similiter addantur figure de 49, scilicet 4 cum 9, erunt 13, de quibus demantur 9, remanebunt 4, que multiplicet cum 1 seruato, erunt 4, que seruet pro pensa, et colligantur figure que sunt in summa multiplicationis, scilicet 1 et 8 et 1 et 3, erunt 13, de quibus demantur 9, remanebunt 4, ut oportet pro pensa remanere.

Procedit hic modus multiplicandi ex his que dixi superius de numero in partes diuiso, et multiplicato in alium quem uis numerum. Nam multiplicatio de 37 in 49,

2

« Item si .... hac descriptione » (fol. 5 recto, lin. 22-28; pag. 9, lin. 1-10).

proba est .1.
9604
98
98

« gradu ut .... ponantur » (fol. 5 recto, lin. 35-38; pag. 9, lin. 20-21, lin. 27).

4900
70
70

fol. 5 verso.

« de 37 .... ostenditur » (fol. 5 verso, lin. 1-6; pag. 9, lin. 28-35).

1813
proba est .4. 37
49

Figure 7: *Examples of multiplications proposed by Fibonacci*[22]

21 *Cordetenus* means "by heart". The expression is referred to the methods of performing the elementary operations, in which the carried numbers are not written but kept in mind. On the operations *cordetenus*, see Smith 1958, 202, note 2.

22 Fibonacci [1228] 1857, 9. We reported calculations below.

Let us consider the rectangular image below:[23]

1813
37
49
49

Leonardo multiplies 7 by 9 obtaining 63. This is the number of units. Hence, let us write 3 and carry 6. Now we have:
7x4 = 28, plus 9x3 = 27, plus the carried 6, gives us the result 61.

This is the number of tens. Therefore let us write 1 and carry 6. Finally, we have: 3x4 = 12 plus the carried 6. This provides the number of hundreds, namely 18.

Consequently 1813 is the final result.

In order to prove that the result is correct, Leonardo used the 9 proof. He had explained, particularly, that

> [...] the residue of every number divided by 9 is the same as the one obtained by the sum of all the digits composing the number [...].[24]

In our case:

3+7=10 gives residue 1, 4+9=13 gives residue 4 and 1x4=4 gives residue 4.

On the other hand, we have:

1+8+1+3=13, with residue 4.

Therefore, according to the 9 proof the result is correct.

The way in which Leonardo presents the multiplication is of high didactic value because the role, the positions and the operations connected with the successive powers of the 10, in function of which the numbers are written in the decimal system, are clearly highlighted. This technique is almost the same as that used in the example of multiplication on Gerbert's *abacus*. The fundamental differences are two: 1) the numbers are written with a pen and no token is necessary; 2) the digit 0 –when it occurs in an operation– is written and there is not a square with an empty place as in the case of Gerbert's *abacus*. Zero is considered and treated in the same way as the natural numbers, at least from an operative point of view. It may seem that these differences are merely formal, but, in fact, they indicate that Fibonacci had fully comprehended the operations on the lines, whereas not all the steps are carried out in Gerbert's *abacus*, even though many of them are.

An example of Extraction of the Square Root from the Liber Abaci

23 This image is the same as the one used by Leonardo.

24 "[...] nam residuum quod remanet ex quovis numero diviso per 9, est summa que ponitur ex additione omnium figurarum facentium ipsum numerum". (Fibonacci [1228] 1857, 8, line 30).

The initial part of the 14th chapter is a complex itinerary that begins with the extraction of the square root of the whole numbers and proceeds explaining many of the rules of calculations with radicals. We will focus on the manner in which Leonardo teaches his eraders how to extract the square root of a number because, from a didactic point of view, this is a real masterpiece. This technique is a novelty for the Western Medieval world and it is quite effective from a didactical point of view.

Fibonacci ([1228] 1857, 353) begins by explaining that the root of numbers with 1 or 2 digits is a number of 1 digit, that of numbers with 3 or 4 digits is a number of 2 digits, that of numbers with 5 or 6 digits is a number with 3 digits and so on. After this clarification, he illustrates the use of his method through a series of examples. The first one concerns the number 743 ([1228] 1857, 353-354)[25]. Therefore, it is required to calculate $\sqrt{743}$. Leonardo proceeds as follows:

		3		6
	14			
7		**4**	3	
		2	7	
		2	7	

Let us consider the biggest number whose square is less than the last digit of 743, namely 7. This number is 2.
He poses 2 under the 4 of 743 (both of them are tens).
After that he multiplies 2 by itself and subtracts the result from 7 obtaining 3.
He writes 3 over 7 and reckons the number 34 (bold in the table).
At this point his reasoning goes this way: he looks for the number *n* having this properties:

[1] *n* multiplied by the double of 2 has to be not larger than 34.
[2] the number $(34-4n)$ posed in front of 3 (the last digit of 743) has to be greater than n^2. In symbols (that are completely missing in Leonardo's treatise) it is ;
$[(34-4n)\cdot 10+3]>n^2$
[3] $[(34-4n)\cdot 10+3]-n^2<2(20+n)$ Considering all these conditions, Leonardo obtains the number 7.
Therefore 27 is the integer part of $\sqrt{743}$.

Since the square root of a number of $2m-1$ or $2m$ digits has m digits, let us divide the digits of the number in groups of two digits starting from the right side. The last group of digits on the left will have 1 digit if the quantity of digits of the number is odd, 2 digits if it is even. In our case we have 7. Therefore the integral part of $\sqrt{7}$ provides the tens of $\sqrt{743}$ and in general the integral part of the root square of the last one–or two digits provides the number of units that are present in the highest power of 10 comprehended in the square

25 In this case we confine ourselves in reconstructing the procedural scheme of Leonardo without reporting the whole page 354 wherein the scheme is presented by Fibonacci. The emphasis in bold in the image is ours.

root. In our case we have 2. Thus, the integral part of $\sqrt{743}$ is a number of the form $(20+n)$ with $0 \leq n \leq 9$. Now, in order to understand the steps above indicated by [1], [2], [3] it is necessary to take into consideration the development of

$$(20+n)^2 = 20^2 + 2 \cdot 20 \cdot n + n^2$$

where $n^2 < 2 \cdot 20 \cdot n < 20^2$

Finally, this property is valid for the square of every binomial $(a+b)$ in which $a>b$.

A reflection on the Property [1]

The property that n multiplied by the double of 2 has to be not larger than 34 means that

$2 \cdot 20 \cdot n$ has to be less than 340,

otherwise $(20+n)^2$ would be greater than 743.

A Reflection on the Property [2]

The property that the number $(34-4n)$ posed in front of 3 has to be greater than n^2 means exactly that

$$343 > 40(= 2 \cdot 20)n + n^2$$

This condition has to be fulfilled because 743 is not a perfect square and hence

$$743-(20+n)^2$$

Or, which is the same, $343 - 2 \cdot 20n - n^2$ must provide a residue.

A Reflection on the Property [3]

The property [3] asserts that $(20+n)^2$ is less than 743, but $(20+n+1)^2$ is greater. Since, if we consider two consecutive squares

m^2 and $(m+1)^2$

their difference is $2m+1$. Thus the property [3] asserts that the difference between

$743 - 20^2 - 2 \cdot 20n$ and n^2 is less than $2(20+n)$,

otherwise, $(20+n+1)^2$ would be less than 743.

From a practical point of view, the easiest solution is to consider the largest number that, multiplied by 4, gives a number less than 34 as a result. This number is 8, but 8 is not good because 23 is less than 64 (property [2] is not fulfilled). After 8, one has to try with 7 and 7 fulfils the properties [2] and [3], too. Therefore, this is the number we were looking for. If 7 had not satisfied one of the conditions, one would have tried 6 and so on. Fibonacci's method can be extended to every number and it can also be used in order to calculate the decimal parts of the square roots.

The Reception of Fibonacci, Abacus schools, Abacus Masters and Abacus Treatises

The reception of Leonardo's work in Western Europe and, particularly, in Italy between the end of the 13th and the first half of the 16th century is an important subject for mathematics, mathematics education, as well as the relation between history of mathematics and civilization. A certain amount of studies on Fibonacci's influence on subsequent mathematics and mathematics education in Italy exists, thanks to the work of Guglielmo Libri Carucci dalla Sommaia (1803-1869).[26] Baldassarre Boncompagni (1821-1894) played a fundamental role in the rediscovery[27] of Fibonacci and in the reconstruction of the life and the influence of the great Pisan mathematician. In the 20th century, owing to the work of scholars such as Paul Louis ver Eecke (1967-1959), Gino Arrighi (1906-2001: Pancanti-Santini 1983) and Ettore Picutti[28] (1979, 1983), many manuscripts of the late Medieval and Early Modern Italian mathematicians were discovered. In this regard, Arrighi's work is fundamental. However, it is largely from the last quarter of the 20th century onwards that a huge and systematic series of works and discoveries about mathematics and mathematics education in Italy in the late Middle Ages and the Early Modern period has been carried out.[29] Hence, today the picture is clearer than in the past, even though there are still uncertainties on important questions, such as the precise influence Fibonacci exerted on the abacus masters.

Abacus Schools and Abacus Masters

It is likely that the first *maestro d'abaco* (*Abacus* teacher, namely a person who taught the calculation with the Hindu-Arabic decimal system) was Fibonacci himself, because the above mentioned document (1241) in which the Municipality (in Pisa) assigned an annual salary to Leonardo probably concerned a salary for a series of services which also included the teaching of the Hindu–Arabic system (Ulivi 2002, chap. I, 2011).
The decimal system and the calculation with the Hindu-Arabic digits were not adopted in a systematic way in the second half of the 13th century. Here we have to take into

26 Libri 1838, II, 1944. As to the influence Fibonacci had on Italian mathematicians, numerous pieces of information are given by Libri in the rest of the second volume of his work.

27 Boncompagni 1852, 1854. He is the editor of all Fibonacci's works (Fibonacci 1857, 1862).

28 Ver Eecke is the translator into French and editor of Fibonacci's *Liber Quadratorum* (Fibonacci 1952).

29 In the last twenty years of the 20th century the university of Siena made an important contribution to the study of mathematics in the Middle Ages, by establishing a *Centre for the study of Medieval Mathematics* which published (under the direction of Franci and Toti Rigatelli) a series of works authored by Medieval mathematicians.

account the fact that at the beginning of the 14th century the *Statuti dell'Arte del Cambio di Firenze* –the rules for the whole conduct of trade– prohibited the use of the decimal system, in order to avoid the risk of frauds in the calculations against those who ignored the decimal system. However, the new way of calculating became progressively more popular and replaced almost completely the old ones. At the beginning, this was the case with the milieu of merchants who were engaged in great business, but progressively the phenomenon was extended to every milieu. This was due basically to the fact that the business became rapidly more extended than in the past and the structure itself of the production and trading companies became more complex. Thus, a precise way to keep the books became important. Book entries were systematically introduced and the trade–networks became more extended and interrelated. This brought about the need for quick and precise calculations (on these topics, see Giusti 2002, chap. 15). Therefore, in many cities *scuole d'abaco* (*Abacus schools*) were established.

Between the second half of the 13th century and the beginning of the 16th century the *Abacus schools* became a mass phenomenon in many Northern and Central Italian cities. Toscana was probably the region in which their number was more conspicuous. Both private and public *Abacus schools* existed.

The public schools were organized and financed by the municipalities. In general, the salaries of the teachers differed with respect to whether they were employed in the public or in the private schools: the teachers of the private schools received higher salaries than those of the public schools (Ulivi 2002, chap. 3).[30] The economic and social *status* of the *Abacus teachers* varied, ranging from that of petty artisans facing economic insecurity to that of relatively rich persons. They can be considered as the first professional mathematicians because they earned their living from their mathematical capabilities and from their teaching abilities (Franci 1988, 183).

Firenze is the city for which the research has been more profound and probably the one in which the *Abacus schools* were most common. Between the second half of the 13th century and the first half of the 16th century, nearly 20 *Abacus schools* operated in Firenze and about 70 Abacus masters. All these schools were private. The most famous one was the *Bottega di Santa Trinita*, active between the first half of the 14th century and the middle of the 15th (Ulivi 2013).

The famous Abacus master Paolo dell'Abaco (1282-1374) probably worked at the *Bottega di Santa Trinita*. Another famous school was the *Scuola di Santa Maria della Scala*, operating during the second half of the 15th century. Here Maestro Benedetto da Firenze

30 Ulivi has recently published a book on the *Abacus schools* in Firenze. This publication is important because it exploits new documents and presents a complete picture of this interesting problem (Ulivi 2013).

(b. 1470) worked. It is likely that he was the most important mathematician among the *Abacus* masters and the teacher, as well, of Leonardo da Vinci (Franci 1988, 190). Some families, such as the Calandri, counted several Abacus masters (Ulivi 2002, chap. 2). Pier Maria Calandri was the teacher of Niccolò Machiavelli (1469-1527) and Filippo Calandri the teacher of Giuliano de' Medici (1453-1478) (Franci 1988, 190).

By contrast, smaller Tuscan cities had public Abacus schools: San Gimignano, operating since 1279, Siena, at the latest since the beginning of the 14th century, Lucca and Pistoia, at the latest since the half of the 14th century, Arezzo, at the latest since the last ten years of the 14th century, Volterra, since the beginning of the 16th century, as well as Prato. In Pisa, after Fibonacci, the first documented presence of Abacus schools dates back to the second half of the 14th century. Outside Toscana, private Abacus schools existed in Bologna at least since 1265, public schools in Verona since 1285, in Savona since 1345, in Perugia since the last twenty years of the 14th century. Modena was an important centre for the public Abacus schools from the beginning of the 15th century, as well as Brescia.

In Venezia the work of the Abacus schools was almost as important as in Firenze and had similar features: the schools were, in general, private. One of the most important was the *Scuola di Rialto* where Luca Pacioli (Pisano 2013, 2009) studied under the maestro Domenico Bragadin (b. first half of the 15th cent.) (Ulivi 2002, chap. I). This suffices to give some idea of the importance and the diffusion of the Abacus schools. The age of the students that frequented the *Abacus schools* was variable. In general, the pupils began their arithmetical instruction at 10-11 and this education lasted for two years. However, there were numerous exceptions.

There are evidences of young pupils who began to attend the *Abacus schools* at 6 and other at 17 (Ulivi 2002, chap. 3). With regard to the social *status* of the pupils, it was rather varied. The reason is that the *Abacus schools* were a phenomenon that concerned most parts of the bourgeoisie: hence, the sons of the artisans, of the men involved –in any manner– in the trading activities, of the public officials and of almost all of the learned or wealthy families frequented these schools. It is estimated that at the end of the 15th century more than 25% of all young people frequented the Abacus schools in Firenze, while, in a small city as Arezzo, the percentage was only 5%. In Venezia, one century later, it was about 40% (Ulivi 2002, chap. 3). The teaching was organized in *mute*.

In general the *mute* were divided as follows:[31]

- The first *muta* called *librettine* contained an introduction to the Hindu–Arabic digits, addition, subtraction and multiplications between integers;
- More than one *muta* –in general three– were dedicated to the division;
- Fractions and their operations;
- Three rule;
- Calculations for money exchange

On the Abacus Treatises

An interesting aspect associated with mathematics education in the Middle Ages is the Italian Abacus treatises. These texts can be considered, at least in part, as handbooks in which the subjects expounded in the Abacus schools were dealt with. The first Abacus treatises date back to the end of the 13th century and this kind of literature was flourishing until the 16th century. In 1980 van Egmond carried out a profound research in the Italian libraries and archives and catalogued about 300 Abacus treatises written (manuscript or printed) in Italy between the late middle ages and the Renaissance (van Egmond 1980, Bagni 1998).

The structure of the Abacus treatises varies according to the epochs and the authors. Some sections are, more or less, the same in every treatise: they concern the introduction of the Hindu–Arabic digits, the four arithmetical operations, the solution of the equations of second degree, some problems connected to the trade and money-exchange, the square roots calculation and some notions of geometry. However, in other respects the Abacus treatises present conspicuous differences because the level of the problems proposed, their explanations and solutions vary from author to author. Some authors present a long series of formulas to solve the equations of second and (in their wrong opinion) of third degree. Sometimes long parts dedicated to astronomy and astrology exist.[32] This is, therefore, a varied literature in which the bounds between didactics of mathematics and mathematical research are not always clear, and in which good intuitions and mistakes handed down from the one author to the other are present.

In the described context, one of the most remarkable aspects concerns algebra, because almost all treatises present solutions to second- and third-degree equations, and often to equations of higher degree, too.[33] As for the second-degree equations, the *Abacus*

31 Cf.: Ulivi 2002, chap. 3.

32 dell'Abaco 1985, Arrighi 1980, 1981, Piochi 1984.

33 dell'Abaco 1964, della Francesca 1970, Gilio 1983, Canacci 1983, Anonimo maestro Lombardo 1983, Gori 1984, Bastiano da Pisa called Bevilacqua (fl. XVI) 1987, Anonimo (sec XIV) 1988, Catellani, Degani and Mantovani 2000, Arrighi 1966-1967, Van Egmond

treatises follow the classical treatment which we find in Abū Ja'far Muhammad ibn Mūsā al-Khwārizmī, better known as al-Khwārizmī (c. 780-c. 850) and Fibonacci (Fibonacci [1228] 1857), but the question becomes interesting from a historical point of view when we consider the equations of third degree, which cannot be reduced to equations of a lower degree. Generally speaking, and using the modern symbolism, the medieval authors refer to the same solution for the equations:

$$ax^3 = bx + c$$

$$ax^3 = bx^2 + c$$

which should be solved by the formula:

$$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} + \frac{b}{2a} \qquad (1)$$

and for the equation:

$$ax^3 = bx^2 + cx + d$$

whose solution should be:

$$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c+d}{a}} + \frac{b}{2a} \qquad (2)$$

Franci,[34] among other scholars, reminds the reader that the first text – as far as we know -, where these rules are expounded is the *Libro di ragioni* by Paolo Gherardi, 1328 (Gherardi, in Arrighi, 1966-1967). Many other abacus treatises mention the same wrong rules, i.e: 1) *Trattato dell'alcibra amuchabile*, by an anonymous author who wrote at the half of the 13th century; 2) Maestro Gilio, *Questioni di algebra*, at the end of the 14th century; 3) the *Regole di geometria e della cosa* by an anonymous in Firenze, written probably around 1460; 4) Raffaele Canacci, *Ragionamenti d'algebra. I problemi.*[35]

The authors did not try to prove the general validity of their formulas or to empirically verify them.

A different and interesting case is that of the *Trattato d'algibra* of an anonymous author who lived in the 14th century (see Anonymous (sec. XIV) (1988)). The author is aware of the fact that the solutions of the third degree equations

$$ax^3 = bx + c;\ ax^3 = bx^2 + c;\ ax^3 + c = bx^2$$

1978, Franci and Toti Rigatelli 1982, 1983,1985.

34 Franci in Gilio 1983, XV.

35 These four works survive in manuscript codices codes. They form part of larger abacus treatises. The titles are given by the modern editors or, in some cases, by the authors themselves. For no. 1 see Anonimo (sec. XIV) 1994; for no. 2 Gilio 1983; for no. 3 Anonimo fiorentino 1992; for no. 4 Canacci 1983.

can be drawn from those of the equation

$$y^3 + py + q = 0 \quad (3)$$

by the replacements

$$x = z - \frac{b}{3a};\ p = \frac{c}{a} - \frac{b^2}{3a^2};\ q = \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3}$$

which is absolutely correct. He does not venture to hypothesize on the formula for the solution of equation (3) and provides approximate solutions to the three equations.

Hence, this author displays a good and profound knowledge of the nature of this problem. Because of this, Raffaella Franci and Marisa Pancanti[36] considered this contribution as the most important for the solution of the third degree equation, before the works of the Italian mathematicians who lived in the 16th century, that is two centuries later.

Another case is that of Maestro Dardi (14th century): in *Aliabraa Argibra* he refers to the equation

$$cx^3 + bx^2 + ax = n \quad 4)$$

and gives the formula

$$x = \sqrt[3]{\left(\frac{a}{b}\right)^3 + \frac{n}{c}} - \frac{a}{b}$$

which is wrong, but provides the right solutions for the cases examined by Dardi:

Table 1: *A solution of equation 4*

Quando le C e llj Ç e llj cubi sono equali al numero, tu dei partire tutta la dequatione per la quantità de cubi, e poi partire la C per li Ç e quello che ne viene reducie a R cubica e quella moltiplicatione giunge sopra al numero, e lla R cuba di quella somma meno lo partimento che ti venne partendo le C per li Ç e tanto vale la C. (Dardi, 2001, 270).	When the unknown [*x*] and the squares [x^2] and the cubes [x^3] (all considered with their coefficients) are posed equal to a given number, divide the whole equation for the cube-coefficient [that is, divide *n* by *c*] and divide the unknown-coefficient by the coefficient of the square-unknown. Elevate the result to the cube [that is, elevate *a/b* to the cube]. After this operation, add the number [*n/c*]. Extract the cubic-root of the result and from this root subtract the division of the unknown-coefficient by the coefficient of the square-unknown. This is the value of the unknown. [The translation is ours].

The problem dealt with by Dardi, in which this equation occurs, concerns – and this is typical of many questions encountered in the abacus treatises – the compound interest (Dardi, 2001, 270-271). The specific problem addressed by Dardi can be transcribed into algebraic terms by the equation

36 Franci-Pancanti, in Anonimo (sec XIV) 1988, xvi-xxi.

$$x^3 + 60x^2 + 1200x = 4000 \qquad 5)$$

As Franci underlines[37], this equation can be written as

$$(x+20)^3 = 12000 \qquad 6)$$

In this case

$$\frac{a}{b} = \frac{1200}{60} = 20\ ;\ \frac{a}{b} = \frac{1200}{60} = 20^3 + 4000 = 12000$$

This is the reason why this particular equation can be solved by Dardi's formula, too, although this formula is wrong as solution to the general third-degree equation.

A similar interesting case is the one of the *Abacus treatise* written by the famous artist Piero della Francesca (della Francesca 1970). Piero della Francesca reported the above mentioned equations (1) and (2) and used them to solve a series of problems of compound interest. He applied the general formula, which is wrong, but notwithstanding this, in the specific cases considered by Piero della Francesca leads to the right solutions. Giusti has shown that these formulas work for the compound interest because they are equivalent –only in this specific case– to the correct formulas for the compound interest given by Fibonacci in the *Liber Abaci*[38] and are independent of the general solution to the third-degree equation (Giusti 1991).

An alternative historiography of the described picture

The classical interpretation concerning the introduction of the Hindu-Arabic digits and positional system in Europe as well as the development of the abacus schools and the abacus treatises is based on the central role of Fibonacci:

1) Fibonacci has introduced –or, at least, if we consider Gerbert and his milieu– reintroduced and spread the Hindu-Arabic digits and the calculation with the positional system;
2) Most of Fibonacci's problems were original productions of his own;
3) The Italian environment was favourable –for economic reasons– to accept Fibonacci's novelties;
4) Abacus schools were established in order to teach what Fibonacci introduced;
5) Abacus treatises were modelled afterthe *Liber Abaci*, even though the language used in them was the Italian vernacular; sometimes they dealt with subjects which were not encountered in the *Liber*; sometimes –on the contrary– they

37 Franci in Dardi 2001, 20.

38 In the *Liber Abaci*, some problems proposed by Fibonacci, concerning compound interest, are particularly significant, considering, also, that Leonardo proposed, for every problem, several methods of solution (Fibonacci 1228, 1857, 267-276).

> did not address problems tackled by Fibonacci; often they provided wrong solutions, which were missing in Leonardo's text.

Arrighi (Arrighi 1987, 10), Giusti (Giusti 2002, 115) and Franci (Franci 2002b, 82) referred to the possibility that the writers of the Italian abacus treatises written in the 14th century knew Arabic sources different from Fibonacci's or that some local mathematical problems-traditions existed; but, after all, these scholars, too, do not basically disagree with the traditional interpretation.

Høyrup's works (see note 4) proposes a different picture concerning both Fibonacci and the sources of the abacus treatises. Essentially, Høyrup expounds these theses, which are based on the idea of an "unknown heritage"[39] and:

1) What Fibonacci expounds in the *Liber Abaci* is in part original, in part drawn from the *Great Arabic Tradition* –the one to which Al-Khwārizmī belongs, too– but in part drawn from a tradition of Arabic derivation, which is different from the *Great one*.
2) What is even more important from a historiographic point of view: this hidden tradition is the one from which the authors of the Italian abacus treatises of the late 13th and of the 14th century mainly drew inspiration. It is their most important source.
3) The numerous references to Fibonacci that we find in the abacus treatises are interpreted by Høyrup as a reference to an author which was regarded as an authority, but who, as a matter of fact, was not always understood by the compilers of the treatises. This means, according to Høyrup, that Fibonacci was not the main source for these authors.

The hidden heritage should have had origin in the Maghreb zone and its spread in Europe should have followed some channels through Andalusia-Catalonia-Provence.

Høyrup also refers to this thesis in all of his writings.

Thus, with regard to the way in which Jacopo da Firenze (beginning of the 14th century) treats some fractions, Høyrup writes:

> This way to put the denominator coincides exactly with the algebraic notation found in Maghreb from the twelfth century onward.[40]
> […] Jacopos' algebra is not derived, neither from Fibonacci nor from the Latin translations of Al-Khwārizmī (or Abū Kāmil) – that much should already be clear.

39 "Unknown Heritage" is the title of Høyrup 2007.

40 Høyrup 2003, 11.

> Its ultimately root in Arabic *al-jabr* are less certain. In consequence, Jacopo's algebra confronts us with a hitherto unknown channel to the Arabic world and its mathematics.[41]

Høyrup also excludes the possibility of an Italian algebraic tradition existing before Jacopus and hence he suggests that the whole tradition of the abacus treatises based upon Jacopo's did not have an autochthonous origin, but, depending on Jacopo's work, derives, in fact from the "unknown heritage" (Maghreb-Andalusia-Catalonia-Provence) (Høyrup 2003, 23-24.) According to Høyrup, Maestro Dardi shares the same tradition as Jacopo (*ivi*, 27). Analyzing the mistakes of an Umbrian abacus treatise written between 1288 and 1290 –in which Fibonacci is quoted several times– Høyrup argues that the writer mentions Fibonacci and some of his problems without having understood Fibonacci's solutions. Furthermore many of the problems proposed by the Umbrian abacus treatise do not exist in Fibonacci. Therefore, Høyrup deduces that Fibonacci is not the real source of this treatise. Thus he thinks that Leonardo introduced the Arabic numerals and the decimal positional system in a mathematical-commercial environment, in which the problems –although their solutions were expressed in Roman numerals– derived form the "unknown heritage" and not from Fibonacci's tradition (Høyrup 2004, 18-19)

The other works of Høyrup develop and deepen this line of argumentation.

Hence, Høyrup has carried out a huge work on this subject and, in the course of years, he has tried to present evidence –based on both linguistic and contextual aspects– which allowed us to realize what the contents of this mathematical heritage were, and in what way it was spread in Europe. Of course, there is no space here to enter into a discussion of the particulars of his work, which are quite interesting and profound.

We underline that Høyrup –in his distinguished research– himself reaches a "pessimistic conclusion" (Høyrup 2010, 19-20) on the reconstruction of the "unknown heritage", because of the lack of direct evidence which support his thesis as to the existence of this heritage. However, the difficulty to prove a particular historiographical thesis does not mean that this thesis is false, considering also the huge amount of evidence by which Høyrup tries to make his thesis plausible. However, some considerations seem to us necessary:

1) In 1784 Adrien-Marie Legendre published his *Éléments de géomètrie*. This book was criticized because of a certain lack of rigour and some logical mistakes. However, Legendre's handbook had great success and was linked to specific educational objectives – which cannot be examined here. There were many editions of Legendre's *Éléments*. A phenomenon called *Legendrism* emerged

41 *Ibidem*.

in the 19th century, that is, the publication of a conspicuous number of geometry-handbooks which relied upon Legendre's work, but which were full of logical and geometrical errors, incomparably beyond the defects of Legendre's text. It was –without any doubt– the reference-model for these books. Therefore, it is absolutely possible that some authors draw from a right source and arrive at wrong conclusions because they do not understand properly the source. This has happened in the 19th century with Legendre, and could have happened in the 13th-14th centuries with Fibonacci, without postulating the existence of any particular heritage. Because Legendre's handbook was not perfect, but far more precise that the successive handbooks *à la Legendre*.[42] Therefore, this argument of Høyrup against Fibonacci's influence on Italian abacus treatises seem to us not quite convincing.

2) The linguistic arguments by Høyrup are profound because they are based on a series of various arguments and evidence: from the way in which the fractions and the ascending continuous fractions are written and interpreted by the various authors, to the occurrences of certain words, to the occurrences of some particular symbols, and so on. It would be superficial and wrong to underestimate this enormous research or to provide an answer in few lines. We confine ourselves to underline that, at that time, a standard mathematical language did not yet exist, and it will not exist for a long period. Inside the same region there were often linguistic differences in the denotation of the same concepts and operations. Thus, if it is a mistake to underestimate the aspects connected to the language, it is also a mistake to overestimate them.
3) With regard to the content; it is clear: the content of many abacus treatises was different from Leonardo's *Liber Abaci*, as Høyrup underlines. However, the problems concerning inheritances, the interest –both simple and compound interest–, the division in part of a certain amount of objects, given certain conditions, are common in all treatises.

Hence, we think that there is no need to think of a different mathematical tradition as a consolidated body of knowledge different from the "Great Arabic Tradition" (see also Pisano and Capecchi 2015, chaps. 2-3) In Europe –as we have already noted more than once– the trade was developing and –in connection with this– the interest in calculating and in solving some problems as those of the item (3). It is not strange that different regions and different authors developed different techniques and languages without any previous tradition being necessary. Fibonacci reached a mathematical level which was far superior to that of his contemporaries, and many of the problem-solving techniques developed by him, and, above all, his general approach to mathematics, were not properly understood.[43]

42 On the "Legendrism" see also Bolondi 2005.

43 For example, Fibonacci perfectly understood that the formula for the second-degree equations could not be applied to that of third degree and realized that some particular

Therefore, on this point it is probably true that Leonardo was an ideal reference-point rather than a real reference-point for most of the authors of the abacus-treatises. It can also be admitted that the problems of the abacus treatises did not have Leonardo as a direct source of inspiration. But, once again, there is no need to postulate a tradition: the problems arise in a specific milieu which is interested in certain operations. While we are convinced that, as far as the spread of the Hindu-Arabic system and of the operation on the lines are concerned, the role of Leonardo was fundamental (albeit, not unique). No evidence can, given the present knowledge, contradict this assertion.

Finally, with regard to the spread of the knowledge (in this case of a wrong knowledge), a quite interesting question for which a final answer does not yet exist is: *From where the wrong formulas of the abacus treatises for the third- and fourth-degree equations derive?*

Certainly not from Leonardo (Pisano 2013). This is sure. But at the moment there is no sure answer. This could be the subject for a future stimulating research.

Conclusion

On Science and its Changing Relationship with Society

The level of the abacus treatises is inferior to that of the *Liber Abaci* because no mathematicians as good as Fibonacci existed in those centuries. At the same time these treatises were written in vernacular Italian since they had to be read by the students and by the merchants and business men who, often, were not skilled in Latin. They were mostly neither foundational nor learned books, but practical texts, even though theoretical parts were not missing. Despite these limitations, at the beginning of the 16th century an original mathematical interest rose among the Abacus masters: it concerned the solution of the equation of a degree higher than the second, in particular, those of the third degree. At the beginning, the difficulty of the question was underestimated and the results were often superficial, but these attempts induced an interest in algebra that was superior than the one Fibonacci had, because he was a great mathematician, but not exactly an algebraist too –even if his contribution was fundamental– but rather a geometer and a number theoretician. The Italian mathematical environment was hence –so to speak– ready to

problems involving equations of a degree higher than the second could be solved by specific techniques which could not be generalized (typical is the problem of the compound interest). Many abacus masters did not grasp this approach –which implies a refined separation between the various situations which can be transcribed into equations as well as a knowledge of the profound differences connoting the problems behind the solutions to equations of different degrees– and reached wrong generalizations with regard to the solutions of the third- and fourth-degree equations.

see the birth of the great algebraic Italian mathematicians of the 16th century whose contributions to the solution of the third- and fourth-degree equations was decisive. On the other hand, in the second half of the 16th century, the calculation with the use of the Hindu-Arabic numerals was consolidated, there was an abundant literature and the need to refer –also ideally– to Fibonacci became less urgent.

The mathematical research began to follow its own *iter*, in the first phase of which algebra played the fundamental role. The consequence of this situation was that the works and the name itself of Fibonacci progressively disappeared and the mathematician who was the most famous and known until the middle of the 15th century, one century later was almost unknown (Bussotti 2009, 48-50). It was not until the middle of the 19th century and Boncompagni that he was rediscovered.

Concluding Remarks

In this paper we have aimed at:

a) Highlighting the relationship between mathematics education and mathematical advanced research in context, because the environment of the *Abacus schools* (and the *masters* and *Treatises* associated with them) was the one which produced the most famous and learned teachers of mathematics in that period and also the one around which some subjects of advanced mathematics –in particular algebra– begun to be dealt with, even though they were not always solved.
b) Pointing out the connections between mathematics education and social structures: at the end of the 12th-beginning of the 13th century Fibonacci succeeded in introducing the operations with the Hindu-Arabic numerals because the Italian maritime republics and, more generally, the Italian city-states developed an economy based on commerce. Hence a quick and precise way of calculating became necessary; Because of this, a plurality of *Abacus schools* emerged between the 13th and the 16th century.

Finally we proposed reflections and examples of interconnected phenomena aimed at showing that an analysis of the relationships between mathematical advanced research and mathematical education (between scientific discoveries and social structure) is a subject that can open new perspectives for understanding the crucial path to the birth of modern science from history of mathematics and science in context.[44]

44 The sequel to the history we have here outlined concerns the solution of the third- and fourth-degree algebraic equations. Once again, the epicenter of this event is Italy. The flourishing of the Italian algebraic researches is contemporary to the Renaissance, a historical–cultural phenomenon, in which, as it is well known, Italian scholars played a prominent role. Very important will be the connections between the general social and cultural environment of Italian society in the 15th and 16th centuries and the scientific and

mathematical discoveries in a series of upcoming works, the first of which will concern the relations between Leonardo da Vinci, Luca Pacioli and Niccolò Tartaglia, which one of us already studies from the standpoint of the history of physics (Pisano 2009, 2011, 2013, 2013 (ed.), Pisano and Bussotti 2013a, 2013b, Pisano and Capecchi 2013, 2015).

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