CHAPTER 10

Mathematics in Medieval Europe

Who wishes correctly to learn the ways to measure surfaces and to divide them, must necessarily thoroughly understand the general theorems of geometry and arithmetic, on which the teaching of measurement . . . rests. If he has completely mastered these ideas, he . . . can never deviate from the truth.

—Introduction to the *Liber embadorum*, Plato of Tivoli's Latin translation of the Hebrew *Treatise on Mensuration and Calculation* by Abraham bar Ḥiyya, 1116[1]

Coming to Pisa in 1225 on orders of the Holy Roman Emperor Frederick II (1194–1250), Leonardo found that his king was interested in mathematics: "After being brought to Pisa by Master Dominick to the feet of your celestial majesty, most glorious prince, I met Master John of Palermo; he proposed to me a question that had occurred to him, pertaining not less to geometry than to arithmetic . . . When I heard recently from a report from Pisa and another from the Imperial Court that your sublime majesty deigned to read the book I composed on numbers [the *Liber Abbaci*] and that it pleased you to listen to several subtleties touching on geometry and numbers, I recalled the question proposed to me at your court by your philosopher. I took upon myself the subject matter and began to compose in your honor this work, which I wish to call *The Book of Squares*. I have come to request indulgence if in any place it contains something more or less than right or necessary, for to remember everything and be mistaken in nothing is divine rather than human; and no one is exempt from fault nor is everywhere circumspect."[2]

10.1 INTRODUCTION TO THE MATHEMATICS OF MEDIEVAL EUROPE

The Roman Empire in the West collapsed in 476 under the onslaught of various "barbarian" tribes. Feudal societies were soon organized in parts of the old empire, and the long process of the development of the European nation-states began. For the next five centuries, however, the general level of culture in Europe was very low. Serfs worked the land and few of the barons could read or write, let alone understand mathematics. In fact, there was little practical need for the subject, because the feudal estates were relatively self-sufficient and trade was almost nonexistent, especially after the Moslem conquest of the Mediterranean sea routes.

Despite the lack of mathematical activity, the early Middle Ages had inherited from antiquity the notion that the **quadrivium**—arithmetic, geometry, music, and astronomy—was required study for an educated man, even in the evolving Roman Catholic culture. Thus, St. Augustine (354–430) had written in his *City of God* that "we must not despise the science of numbers, which, in many passages of Holy Scripture, is found to be of eminent service to the careful interpreter. Neither has it been without reason numbered among God's praises: 'Thou hast ordered all things in number, and measure, and weight.'"[3] Yet the only texts available for the study of these subjects were brief introductions, especially those by the Roman scholar Boethius (480–524) and the seventh-century bishop, Isidore of Seville (560–636). Thus, the outline of the mathematical quadrivium was in place, but it was only a shell, nearly devoid of substance.

Virtually the only schools in existence were connected with the monasteries, many of which were founded by monks from Ireland, the first country not originally part of the Roman Empire to adopt Christianity. While much of continental Europe was in turbulence, these monks copied Greek and Latin manuscripts and thus preserved much ancient learning. Students from all over Europe came to study there. Then, from the sixth to the eighth centuries, missionaries went out from Ireland to the continent to establish new centers of learning from which, several centuries later, new intellectual developments eventually sprung forth.

Even in the earliest part of the Middle Ages, however, there was a significant mathematical problem to be considered: the determination of the calendar. In particular, the Church debated whether Easter should be determined using the Roman solar calendar or the Jewish lunar calendar. The two reckonings could be reconciled, but only by those with some mathematical knowledge. Thus, Charlemagne, even before his coronation in 800 as Holy Roman Emperor, formally recommended that the mathematics necessary for Easter computations be part of the curriculum in Church schools.

To help him in establishing more schools, Charlemagne brought in Alcuin of York (735–804) as his educational adviser. Alcuin, who had studied with an Irish teacher and was assisted in Charlemagne's court by several Irish clerics, generally sent to England and Ireland when he needed books. We do not have much direct information about Alcuin's knowledge of mathematics, but a collection of fifty-three arithmetical problems from his time, entitled *Propositiones ad acuendos juvenes* (*Propositions for Sharpening Youths*), is generally attributed to him. The problems of the collection often require some ingenuity for solving, but do not depend on any particular mathematical theory or rules of procedure.

FIGURE 10.1
Gerbert d'Aurillac, Pope Sylvester II

In the tenth century, a revival of interest in mathematics began with the work of Gerbert d'Aurillac (945–1003), who became Pope Sylvester II in 999 (Fig. 10.1). In his youth, Gerbert studied in Spain, where he probably learned some of the mathematics of the Moslems. Later, under the patronage of Otto II, the Holy Roman Emperor, Gerbert reorganized the cathedral school at Rheims and successfully reintroduced the study of mathematics. Besides teaching basic arithmetic and geometry, Gerbert dealt with the mensuration rules of the Roman surveyors and the basics of astronomy. He also taught the use of a counting board, divided into columns representing the (positive) powers of 10, in each of which he would place a single counter marked with the western Arabic form of one of the numbers 1, 2, 3, . . . , 9. Zero was represented by an empty column. Gerbert's work represents the first appearance in the Christian West of the Hindu-Arabic numerals, although the absence of the zero and the lack of suitable algorithms for calculating with these counters showed that Gerbert did not understand the full significance of the Hindu-Arabic system.

Despite the limited mathematical sources available to Europeans at the turn of the millennium, scholars did know that there was an ancient tradition in mathematics due to the Greeks, but it was virtually inaccessible to them at the time. This heritage, as well as a portion of the mathematics developed in the Islamic world, was only brought into western Europe through the work of translators. European scholars discovered the major Greek scientific works (primarily in Arabic translation) beginning in the twelfth century and started the process of translating these into Latin. Much of this work was accomplished at Toledo in Spain, which at the time had only recently been retaken by the Christians from the former Moslem rulers. Here could be found repositories of Islamic scientific manuscripts as well as people straddling the two cultures. In particular, there was a flourishing Jewish community, many of whose members were fluent in Arabic. The translations then were often made in two stages, first by a Spanish Jew from Arabic into Spanish, and then by a Christian scholar from Spanish into Latin. The list of the translations of major mathematical works (with their dates) is fairly extensive (Sidebar 10.1).

Among the earliest of the translating teams were John of Seville and Domingo Gundisalvo, who were active in the first half of the twelfth century. John was born a Jew, his original name probably being Solomon ben David, but converted to Christianity, while Gundisalvo was a philosopher and Christian theologian. The most important of their mathematical translations was of an elaboration of al-Khwārizmī's work on arithmetic. They also translated a large number of astronomical works, including commentaries on the work of Ptolemy, and numerous medical and philosophical works.

A contemporary of John of Seville was Adelard of Bath (1075–1164), who was born in Bath and spent much of his early years traveling in France, southern Italy, Sicily, and the Near East, the latter two places in particular having many Arabic treatises available. Adelard was responsible for the first translation from the Arabic of Euclid's *Elements*. He also translated the astronomical tables of al-Khwārizmī in 1126. This translation contains the first sine tables available in Latin as well as the first tangent tables, the latter having been added to al-Khwārizmī's work by an eleventh-century editor. Another Englishman, Robert of Chester, who lived in Spain for several years, translated the *Algebra* of al-Khwārizmī in 1145, thus introducing to Europe the algebraic algorithms for solving quadratic equations. Interestingly enough, in the same year, Plato of Tivoli translated from the Hebrew the *Liber*

SIDEBAR 10.1 *Translators and Their Translations*

James of Venice (fl. 1128–1136)

- *Topics, Prior Analytics, Posterior Analytics* of Aristotle

Adelard of Bath (fl. 1116–1142)

- *Astronomical Tables* of al-Khwārizmī
- *Elements* of Euclid
- *Liber ysagogarum Alchorismi*, the arithmetical work of al-Khwārizmī

John of Seville and Domingo Gundisalvo (fl. 1135–1153)

- *Liber alghoarismi de practica arismetrice*, an elaboration of al-Khwārizmi's *Arithmetic*

Plato of Tivoli (fl. 1134–1145)

- *Spherica* of Theodosius (c. 100 BCE)
- *De Motu Stellarum* of al-Battānī, which contains important material on trigonometry
- *Measurement of a Circle* of Archimedes
- *Liber embadorum* of Abraham bar Ḥiyya

Robert of Chester (fl. 1141–1150)

- *Algebra* of al-Khwārizmī
- Revision of al-Khwārizmī's astronomical tables for the meridian of London

Gerard of Cremona (fl. 1150–1185)

- *Posterior Analytics* of Aristotle
- *De Sphaera Mota* of Autolycus
- *Elements* of Euclid
- *Data* of Euclid
- *Measurement of a Circle* of Archimedes
- *Spherica* of Theodosius
- *Almagest* of Ptolemy
- *De Figuris Sphaericis* of Menelaus
- *Algebra* of al-Khwārizmī
- *Elementa Astronomica* by Jābir ibn Aflaḥ

Wilhelm of Moerbeke (fl. 1260–1280)

- *On Spirals* of Archimedes
- *On the Equilibrium of Planes* of Archimedes
- *Quadrature of the Parabola* of Archimedes
- *Measurement of a Circle* of Archimedes
- *On the Sphere and Cylinder* of Archimedes
- *On Conoids and Spheroids* of Archimedes
- *On Floating Bodies* of Archimedes

Note: This listing contains works whose translation can definitely be attributed to a given translator. There are Latin translations of other works known to have been made in the twelfth and thirteenth centuries, including parts of Apollonius's *Conics* and the *Algebra* of Abū Kāmil, whose translators are currently unknown.

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embadorum (*Book of Areas*) by the Spanish-Jewish scholar Abraham bar Ḥiyya, a work that also contained the Islamic rules for solving quadratic equations.

The most prolific of all the translators was Gerard of Cremona (1114–1187), an Italian who worked primarily in Toledo and is credited with the translation of more than 80 works. Undoubtedly, not all of these are due to him alone. It is known that one of his assistants was Galippus, a Spanish Christian who had been allowed to practice Christianity under Moslem rule, but the names of his other assistants have been lost to history. Among Gerard's works was a new translation of Euclid's *Elements* from the Arabic of Thābit ibn Qurra and the first translation of Ptolemy's *Almagest* from the Arabic in 1175.

By the end of the twelfth century, then, many of the major works of Greek mathematics and a few Islamic works were available to Latin-reading scholars in Europe. During the next centuries, these works were assimilated and new mathematics began to be created by the Europeans themselves. It is well to note, however, that some Spanish-Jewish scholars had earlier read the Arabic works in the original and had produced works on their own, in Hebrew.

During the twelfth century, in fact, the cultural exchange among the three major civilizations of Europe and the Mediterranean basin, the Jewish, Christian, and Islamic, was very intense. The Islamic supremacy of previous centuries was on the wane, and the other two were gaining strength. By the end of the next century, the genius of western Christendom had manifested itself, while various physical limitations on the lives of the Jews began to lessen the Jewish contribution.

This chapter will discuss both Jewish and Christian contributions of the twelfth through the fourteenth centuries. We will first consider geometry and trigonometry, next developments in combinatorics, next the algebra that grew out of the introduction of Islamic algebra into Europe, and finally, some of the mathematics of kinematics that stemmed from the study of Aristotle's works in the medieval universities.

10.2 GEOMETRY AND TRIGONOMETRY

Euclid's *Elements* was translated into Latin early in the twelfth century. Before then, of course, Arabic versions were available in Spain. And so, when Abraham bar Ḥiyya (d. 1136) of Barcelona wrote his *Hibbur ha-Meshihah ve-ha-Tishboret* (*Treatise on Mensuration and Calculation*) in 1116 to help French and Spanish Jews with the measurement of their fields, he began the work with a summary of some important definitions, axioms, and theorems from Euclid. Not much is known of the life of Abraham bar Ḥiyya, but from his Latin title of *savasorda*, a corruption of the Arabic words meaning "captain of the bodyguard," it is likely that he had a court position, probably one in which he gave mathematical and astronomical advice to the Christian monarch.

10.2.1 Abraham bar Ḥiyya's *Treatise on Mensuration*

Like most of those who dealt with geometry over the next few centuries, Abraham was not so much interested in the theoretical aspects of Euclid's *Elements* as in the practical application of geometric methods to measurement. Nevertheless, he took over the Islamic tradition of proof, absorbed from the Greeks, and gave geometric justifications of methods for solving the algebraic problems he included as part of his geometrical discussions. In particular, Abraham included in his work the major results of *Elements* II on geometric algebra and used them to demonstrate methods of solving quadratic equations. In fact, Abraham's work was the first in Europe to give the Islamic procedures for solving such equations.

For example, Abraham posed the question, "If from the area of a square one subtracts the sum of the (four) sides and there remains 21, what is the area of the square and what is the length of each of the equal sides?"[4] We can translate Abraham's question into the quadratic equation $x^2 - 4x = 21$, an equation he solves in the familiar way by halving 4 to get 2, squaring this result to get 4, adding this square to 21 to get 25, taking the square root to get 5, and then adding that to the half of 4 to get the answer 7 for the side and the answer 49 for the area. Abraham's statement of the problem was not geometrical, in that he wrote of subtracting a length (the sum of the sides) from an area. But in his geometric justification, he restated the problem to mean the cutting off of a rectangle of sides 4 and x from the original square of unknown side x to leave a rectangle of area 21. He then bisected the side of length 4 and applied *Elements* II–6 to justify the algebraic procedure. Thus, Abraham evidently had learned his algebra not from al-Khwārizmī (whose *Algebra* was translated into Latin in the same year as Abraham's work), but from an author such as Abū Kāmil, who used Euclidean justifications. Abraham similarly presented the method and Euclidean proof for examples of the two other Islamic classes of mixed quadratic equations, $x^2 + 4x = 77$ and $4x - x^2 = 3$. In the latter case, he gave both positive solutions. Abraham also solved such quadratic problems as the systems $x^2 + y^2 = 100$, $x - y = 2$, and $xy = 48$, $x + y = 14$.

Abraham's most original contribution, however, is found in his section on measurements in circles. He began by giving the standard rules for finding the circumference and area of a circle, first using 3 1/7 for π but then noting that if one wants a more exact value, as in dealing with the stars, one should use $3\frac{8\frac{1}{2}}{60} (= 3\frac{17}{120})$. Curiously, in the Hebrew version of the text, but not in the Latin, there is a justification of the area formula $A = \frac{C}{2}\frac{d}{2}$ by use of indivisibles. Namely, one thinks of the circle as made up of concentric circles of indivisible threads (Fig. 10.2). If one then slices this circle from the center to the circumference and unfolds it into a triangle, the base of the triangle is the original circumference and the height is the radius. The area formula follows immediately.

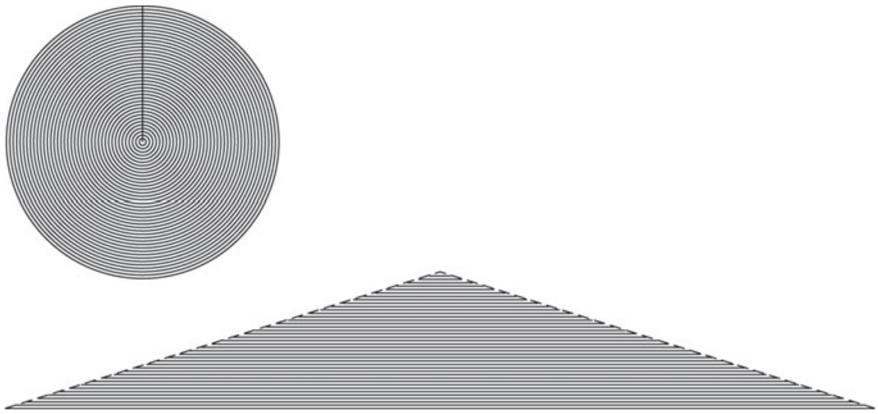


FIGURE 10.2
Circle unfolded into a triangle

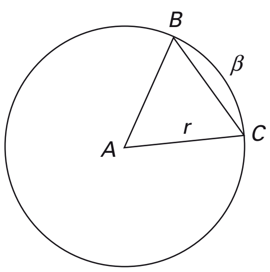


FIGURE 10.3
Area of segment $B\beta C$ = Area of sector $AB\beta C$ − Area of triangle ABC; Area of sector $= r\frac{\beta}{2}$

To measure areas of segments of circles, Abraham noted that one must first find the area of the corresponding sector by multiplying the radius by half the length of the arc (Fig. 10.3). One then subtracts the area of the triangle formed by the chord of the segment and the two radii at its ends. But how does one calculate the length of the arc, assuming one knows the length of the chord? Abraham's answer is, by the use of a table relating chords and arcs. And so for the first time in Europe there appeared what one can call a trigonometric table (Fig. 10.4). Unlike the table of sines of al-Khwārizmī, which appeared in Latin translation shortly after Abraham's book and which used degrees to measure arcs and a circle radius of 60, Abraham's table was a table of arcs to given chords using what seemed to Abraham more convenient measures. Namely, he used a radius of 14 parts, so the semicircumference would be integral (44), and then gave the arc (in parts, minutes, and seconds) corresponding to each integral value of the chord from 1 to 28. So to determine the length of the arc of a segment of a circle, given the chord s and the distance h from the center of the chord to the circumference, Abraham first determined the diameter d of the circle by the formula

FIGURE 10.4
Arc-chord table of Abraham bar Ḥiyya

Partes Cordatum	Arcus		
	Partes	Min.	Sec.
1	1	0	2
2	2	0	8
3	3	0	26
4	4	0	55
5	5	1	44
6	6	2	54
7	7	4	42
8	8	7	11
9	9	9	56
10	10	13	42
11	11	18	54
12	12	24	38
13	13	31	9
14	14	40	0
15	15	50	10
16	17	2	16
17	18	16	36
18	19	33	27
19	20	53	26
20	22	17	10
21	23	45	6
22	25	19	24
23	27	0	0
24	28	49	56
25	31	26	37
26	33	20	52
27	36	27	32
28	44	0	0

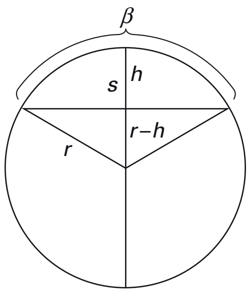


FIGURE 10.5
Length of arc $\beta = \frac{d}{28}$ arc-chord $\left(\frac{28}{d}s\right)$, where $d = 2r = \frac{s^2}{4h} + h$

$d = s^2/4h + h$ (Fig. 10.5). Then he multiplied the given chord by $\frac{28}{d}$ (to convert to a circle of diameter 28), consulted his table to determine the corresponding arc α, and multiplied α by $\frac{d}{28}$ to find the actual arc length.

10.2.2 Practical Geometries

Abraham's Hebrew text was one of the earliest of many practical geometrical works to appear in medieval Europe. An early Latin one appeared in the 1120s, probably written by Hugh of St. Victor (1096–1141), a theologian and master of the abbey of St. Victor in Paris. This text, designed for surveyors, is on a much simpler level than Abraham's. Apparently, knowledge of trigonometry had not yet reached Paris nor was there any mention of Euclid in Hugh's work. But Hugh did make use of the astrolabe, the sighting device developed by Islamic astronomers from earlier Greek models and brought through Spain into western Europe. Thus, Hugh's methods of measurement involved the use of the alidade, an altitude-sighting device attached to the astrolabe, which enabled one to measure the ratio of height to distance of an object sighted (Fig. 10.6). If this ratio r is known, and the distance d of the object is also known,

FIGURE 10.6

Astrolabe with alidade OA. One holds the line OB horizontal and sights the distant object along OA. Then r gives the ratio of the height to the distance of that object.

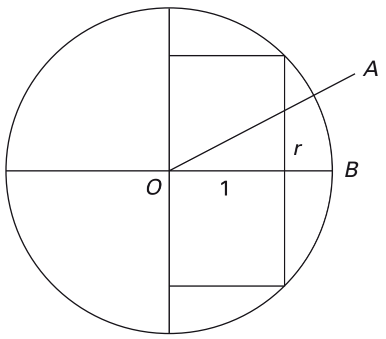
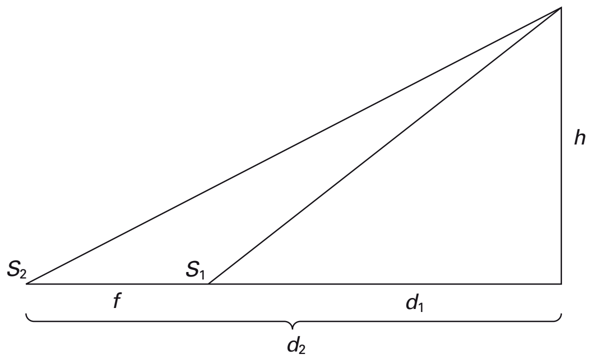


FIGURE 10.7

Measuring height of a distant object using two sightings according to Hugh of St. Victor



then the height h is given by $h = rd$. Like his predecessors in India, China, and the Islamic world, Hugh also knew that it is not always possible to measure the distance d of a distant object. In that case, two measurements were needed (Fig. 10.7). At point S_1, one finds the ratio r_1 of height h to distance d_1, while at point S_2, one finds the ratio r_2 of h to d_2. It then follows that $d_2 = (r_1/r_2)d_1$. But since $d_2 - d_1 = f$ can be measured, Hugh could calculate d_1 as

$$d_1 = \frac{f}{\frac{r_1}{r_2} - 1}$$

and then evaluate h by $h = r_1 d_1$.[5]

By late in the twelfth century, however, trigonometry and knowledge of Euclid had reached Paris, as exemplified in the anonymous practical geometry generally known by the first three words of the manuscript, *Artis cuiuslibet consummatio* (*The Perfection of Any Art*). This work, originally written in Latin but translated into French in the thirteenth century, opens with a rather poetic introduction:

> The perfection of any art, seen as a whole, depends on two aspects: theory and practice. Anyone deprived of either of these aspects is labeled semiskilled. Truly the modern Latins . . . [by] neglecting the practice fail to reap where they sowed the richest fruits as if picking a spring flower without waiting for its fruit. What is sweeter when once the qualities of numbers have been known through arithmetic than to recognize their infinite dispositions by subtle calculation,

> the root, origin, and source necessarily available for every science? What is more pleasant when once the proportion of sounds has been known through music than to discern their harmonies by hearing? What is more magnificent when once the sides and angles of surfaces and solids have been proved through geometry than to know and investigate exactly their quantities? What is more glorious or excellent when once the motion of the stars has been known through astronomy than to discover the eclipses and secrets of the art? We prepare for you therefore a pleasant treatise and delightful memoir on the practice of geometry so that we may offer to those who are thirsty what we have drunk from the most sweet source of our master.[6]

To be truly educated, the author seems to be saying, one not only must study the theoretical aspects of the quadrivium, but also must understand how these subjects are used in the real world. *Artis cuiuslibet consummatio* intends to show, then, the practical aspects of one of the quadrivial subjects, namely, geometry.

The book is divided into four parts: area measurement, height measurement, volume measurement, and calculation with fractions. The last section is designed to help the reader with the computations necessary in the earlier parts. The first part, on areas, begins with the basic procedures for finding the areas of triangles, rectangles, and parallelograms, most of which are justified by an appeal to Euclidean propositions. The author followed this with a section on the areas of various equilateral polygons, all of the formulas for which are incorrect. Instead of being formulas for areas of pentagons, hexagons, heptagons, and so on, of side n, the formulas are always those for the nth pentagonal, hexagonal, heptagonal number. For example, the procedure given for finding the area of a pentagon of side n amounts to using the formula

$$A = \frac{3n^2 - n}{2}.$$

The author may well have been influenced by the material on figurate numbers derived from the work of Nicomachus.

The section of the book on heights showed the author's knowledge of trigonometry. For example, the procedure for measuring the altitude of the sun using the shadow of a vertical gnomon of length 12 is given: "Let the shadow be multiplied by itself. Let 144 be added to the product. Let the root of the whole sum be taken. And then let the shadow be multiplied by 60. Let the product be divided by the root found. The result will be the sine; let its arc be found. Let the arc be subtracted from 90; the remainder will be . . . the altitude of the sun."[7] Namely, if the shadow is designated by s, the altitude α is given by

$$\alpha = 90 - \text{arcSin}\left(\frac{60s}{\sqrt{s^2 + 144}}\right),$$

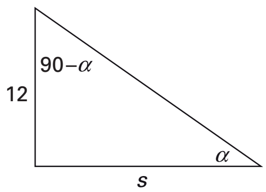


FIGURE 10.8
The calculation of the altitude of the sun given the shadow, and conversely, from the *Artis cuiuslibet consummatio*: $\alpha = 90 - \arcsin\left(\frac{60s}{\sqrt{s^2+144}}\right)$; $s = \frac{12\sin(90-\alpha)}{\sin\alpha}$

where, as in most of the Islamic trigonometric works, the Sines were computed using a radius of 60 (Fig. 10.8). Similarly, the author calculated the shadow from the altitude by using

$$s = \frac{12\ \text{Sin}(90 - \alpha)}{\text{Sin}\ \alpha}.$$

These two problems demonstrate that the author knew the use of a table of sines but probably did not know of the tables of cosines, tangents, or cotangents, even though these had already been developed in the Islamic world. It was only the earliest of the Hindu and Islamic improvements on Greek work that were available.

For surveying, the author returned to the ancient methods. To measure the height of a tower, not only did he not use trigonometric methods, he reverted to probably the oldest (and simplest) method available: "Wait until the altitude of the sun is 45 degrees . . . ; then the shadow lying in the plane of any body will be equal to its body."[8] If the tower is inaccessible, the author used the ancient methods requiring two sightings similar to those in the Chinese and Indian sources. As in almost all of the Indian, Islamic, and medieval European sources, even when trigonometric methods were known, they were applied solely to heavenly triangles, not to earthly ones.

These two twelfth-century Latin geometries give us an idea of the state of geometrical knowledge in northern Europe of the time. Greek geometrical traditions were just beginning to be reestablished, but practical geometrical methods, also dating to ancient times and not all strictly correct, were used for actually computing geometrical quantities of use in daily life. In southern Europe, however, the Islamic influence was stronger and Euclidean traditions of proof are more in evidence, as in the work of Abraham bar Ḥiyya. Another example is provided by the geometrical work of one of the first Italian mathematicians, Leonardo of Pisa (c. 1170–1240).

10.2.3 Leonardo of Pisa's *Practica Geometriae*

Leonardo's *Practica geometriae* (1220) is more closely related to the work of Abraham bar Ḥiyya than to the *Artis cuiuslibet consummatio* or the work of Hugh of St. Victor. In fact, some of the sections appear to be taken almost directly from the *Liber embadorum*. Leonardo's work is, however, somewhat more extensive. As in the earlier book, Leonardo began with a listing of various definitions, axioms, and theorems of Euclid, including especially the propositions of Book II. So in his section on measuring rectangles, in which he includes the standard methods for solving quadratic equations, he was able to quote Euclid in justification of his procedures. He provided more examples than Abraham, including equations in which the coefficient of the square term is greater than 1. For example, to solve the equation three squares and four roots equal 279 ($3x^2 + 4x = 279$), he divided by 3 and reduced the equation to $x^2 + 1\frac{1}{3}x = 93$ before applying the standard method. Also, many of his problems involve the diagonal of a rectangle and thus deal with the sums of the squares of the sides.

Leonardo, again like Abraham, wrote a section on circles in which he quoted the standard $22/7$ for π. But Leonardo, in addition, showed how to calculate this value by the procedure of Archimedes. He found that the ratio of the perimeter of a 96-sided polygon circumscribed about a circle to the diameter of the circle is 1440 to 458 1/5, and the ratio of the perimeter of an inscribed 96-sided polygon to the diameter is 1440 to 458 4/9. Noting that 458 1/3 is approximately halfway between 458 1/5 and 458 4/9, he asserted that the ratio of circumference to diameter is close to $1440 : 458\ 1/3 = 864 : 275$. Because $864 : 274\ 10/11 = 3\ 1/7 : 1$, Leonardo had rederived the Archimedean value.

Leonardo also calculated areas of segments and sectors of circles. To do this, he too needed a table of arcs and chords. Strangely enough, although he defined the Sine of an arc in the standard way, he did not give a table of Sines, but one of chords, and in fact reproduced the Ptolemaic procedure for determining the chord of half an arc from that of the whole arc. His chord table, though, was not Ptolemaic. In fact, it may well be original to Leonardo because it is based on a radius of 21. Like the value 14 of Abraham, this was chosen so the semicircumference of the circle is integral, but unlike Abraham's table, this table is a direct chord table (Fig. 10.9). For each integral arc from 1 to 66 rods (and also from 67 to 131), the table gives the corresponding chord, in the same measure, with fractions of the rods not in sixtieths, but in the Pisan measures of feet (6 to the rod), unciae (18 to the foot), and points (20 to the uncia). Leonardo then demonstrated how to use the chord table to calculate arcs to chords in circles of radius other than 21.

Like Abraham bar Ḥiyya, Leonardo used the table of chords only to calculate areas of circular sectors and segments. When, later in the same chapter, he calculated the lengths of the sides and diagonals of a regular pentagon inscribed in a circle, he did not use what seems to us the obvious method of consulting his table of chords. He returned to Euclid and quoted appropriate theorems from Book XIII relating the sides of a hexagon, pentagon, and decagon to enable him actually to perform the calculations. And toward the end of the book, when he wanted to calculate heights, again he did not use trigonometry. He used the old methods of similar triangles, starting with a pole of known height to help sight the top of the unknown object, then measuring the appropriate distances along the ground.

10.2.4 Trigonometry

That trigonometry in the medieval period was not used to measure earthly triangles is further demonstrated by two fourteenth-century trigonometry works, one by the Englishman Richard of Wallingford (1291–1336) and the other by the French Jew Levi ben Gerson (1288–1344). Yet both of these texts had something new, especially in the methods of calculating accurate tables.

Richard of Wallingford was a monk who spent the final nine years of his life as the abbot at St. Albans monastery. The *Quadripartitum*, a four-part work on the fundamentals of trigonometry, was written while he was still a student at Oxford, probably around 1320. Perhaps 10 years later, Richard revised and shortened this work in another treatise entitled *De Sectore*. The goal of both works, like that of most texts on trigonometry, was to teach the methods required for the solution of problems in spherical trigonometry, which in turn was required for astronomy. It appears that the chief source of the *Quadripartitum* was the *Almagest* of Ptolemy, modified to incorporate the Hindu Sines in addition to the more ancient chords. But by the time Richard revised the work, he had become familiar with Jābir's trigonometry. In fact, in his section on spherical trigonometry, he presented virtually the whole of Jābir's treatment right after Ptolemy's version based on the theorem of Menelaus.

Richard's treatment of the theorem of Menelaus, both in its plane and spherical versions, was extremely detailed. Because this theorem is concerned with ratios among the various sides in the Menelaus configuration, Richard needed first to consider the basics of the theory of proportions. His study of proportions is closely related to the work of several contemporaries in the universities and will be considered in Section 10.5.1. Here we only note that in his treatment of Menelaus's theorem, Richard considered all the possible cases of the Menelaus configuration and proved the result anew each time. While modern readers might consider his work tedious, he evidently felt that such detail was necessary for the less mathematically experienced readers for whom he was writing. One also sees here, as well as in the beginning sections of the book on the basic results of plane trigonometry, Richard's commitment to strictly Euclidean rigor of argument as he exhausts all the cases. Recall that even though mathematical knowledge was at a low ebb during the early Middle Ages, the basic notion of

FIGURE 10.9

Chord table of Leonardo of Pisa

Arcus pertice	Arcus pertice	Corde pertice	Ar pedes	Cuu vncie	M puncta	Arcus pertice	Arcus pertice	Corde pertice	Ar pedes	C V vncie	VM puncta
1	131	0	5	17	17	34	98	30	2	6	17
2	130	1	5	17	13	35	97	31	0	8	5
3	129	2	5	17	4+	36	96	31	4	8	7
4	128	3	5	17	2	37	95	32	2	5	15
5	127	4	4	12	10	38	94	33	0	1	9
6	126	5	5	16	7+	39	93	34	3	13	0
7	125	6	5	14	5	40	92	35	1	4	15
8	124	7	5	12	9	41	91	35	4	12	10
9	123	8	5	8	16	42	90	36	2	0	0
10	122	9	5	7	8	43	89	36	5	3	5
11	121	10	5	4	2	44	88	37	2	4	6
12	120	11	4	17	18	45	87	37	5	3	2
13	119	12	4	13	6	46	86	38	1	17	15
14	118	13	4	7	16	47	85	38	4	12	13
15	117	14	4	1	0	48	84	38	1	4	0
16	116	15	3	11	18	49	83	39	3	11	15
17	115	16	3	3	12	50	82	39	5	17	2
18	114	17	2	12	8	51	81	40	2	2	1
19	113	18	2	0	15	52	80	40	4	2	10
20	112	19	1	8	12	53	79	40	0	0	11
21	111	20	0	13	18	54	78	40	1	14	5
22	110	21	0	0	0	55	77	41	3	7	8
23	109	21	5	2	16	56	76	41	4	16	2
24	108	22	4	4	5	57	75	41	0	4	12
25	107	23	3	4	8	58	74	41	1	8	1
26	106	24	2	3	2	59	73	41	2	9	0
27	105	25	1	0	6	60	72	41	3	7	14
28	104	25	5	16	2	61	71	41	4	9+	2
29	103	26	4	8	0	62	70	41	4	15	10
30	102	27	3	0	3	63	69	41	5	6	9
31	101	28	1	9	7	64	68	41	5	12	17
32	100	28	5	16	4	65	67	41	5	6	14
33	99	29	4	3	9	66	66	42	0	0	0

BIOGRAPHY

Leonardo of Pisa (c. 1170–1240)

Leonardo, often known today by the name Fibonacci (son of Bonaccio) given to him by Baldassarre Boncompagni, the nineteenth-century editor of his works, was born around 1170. His father was a Pisan merchant who had extensive commercial dealings in Bugia on the North African coast (now Bejaia, Algeria). Leonardo spent much of his early life there learning Arabic and studying mathematics under Moslem teachers. Later he traveled throughout the Mediterranean, probably on business for his father. At each location, he met with Islamic scholars and absorbed the mathematical knowledge of the Islamic world. After his return to Pisa in about 1200, he spent the next 25 years writing works in which he incorporated what he had learned. The ones that have been preserved include the *Liber abbaci* (1202, 1228), the *Practica geometriae* (1220), and the *Liber quadratorum* (1225). Leonardo's importance was recognized both at the court of Frederick II, as noted in the opening story, and also in the city of Pisa, which in 1240 granted him a yearly stipend in thanks for his teaching and other services to the community.

a mathematical proof survived and was reinvigorated, as, for example, by Richard, once the need for more mathematics had established itself.

Although much of Richard's work was derived from earlier trigonometries, he did present a new method of calculating Sin 1°, the value that determined the accuracy of the Sine tables. Thus, after considering both Ptolemy's method from the *Almagest* and the method of Abū al-Wafā', he extended the latter to smaller and smaller arcs. Namely, beginning with the Sine of $\frac{3}{16}^\circ$, calculated by the half-angle formula, he suggested continuing to use that formula to find the Sines of $\frac{3}{32}^\circ$ and $\frac{3}{64}^\circ$. The latter enables one to determine, by the sum formula, $\text{Sin}\,\frac{63}{64}^\circ = \text{Sin}(\frac{3}{64}+\frac{15}{16})^\circ$. Similarly, one can find $\text{Sin}(1-\frac{1}{256})^\circ$ and $\text{Sin}(1-\frac{1}{1024})^\circ$ and "proceed in this way even to the 9000th part of a degree, or even to the infinitely small, if by working minutely you wish to do so."[9]

The trigonometrical work of Levi ben Gerson was roughly contemporaneous with the *Quadripartitum*. It formed part of an astronomical treatise that in turn formed part of a major philosophical work, *Sefer Milḥamot Adonai* (*Wars of the Lord*). Levi's trigonometry was based chiefly on Ptolemy, though again, like Richard, Levi generally used Sines rather than chords. Also, like Richard, Levi spent some time dealing with accuracy of his tables. In particular, he noted that tables with intervals of 1° have errors of about 15 minutes of arc when one uses linear interpolation to find arcs corresponding to given Sines, if the arcs are close to 90°. And this large an error was unacceptable. Hence, Levi determined his own tables in steps of $\frac{1}{4}^\circ$.

Levi's main departure from Ptolemy, and also from Richard, is that he gave detailed procedures for solving plane triangles. He first presented the standard methods for solving right triangles and then proceeded to general triangles. In the case where three sides are known, Levi solved the triangle by dropping a perpendicular from one vertex to the opposite side (or opposite side extended), and then applying the version of the law of cosines of *Elements* II–12 and II–13. The same method works also where two sides and the included angle are known. For the case where two sides and the angle opposite one of them are known, Levi used (with proof) the law of sines. He did not, however, mention the possible ambiguity of this case. Of course in any particular problem, one of the unknown angles is required to be acute or obtuse, so a single solution of the triangle can be found. Finally, Levi noted that the case where two angles and a side are known can also be solved using the law of sines.

Certainly, Levi's methods were not new. Although his procedures were somewhat different from those of Jābir, the methods were available in other Islamic trigonometries. Nevertheless, Levi's brief treatise provided one of the earliest treatments of the basic methods for solving plane triangles available in Europe. But as in the Islamic works and the practical geometry texts, the methods Levi presented were used only for solving astronomical triangles, never for solving earthly ones.

10.3 COMBINATORICS

We have already discussed the interest in combinatorics in Indian and Islamic sources. In medieval Europe, there was also interest in such questions, primarily in the Jewish community. The earliest Jewish source on this topic seems to be the mystical work *Sefer Yetsirah* (*Book of Creation*), written sometime before the eighth century and perhaps as early as the second century. In it the unknown author calculated the various ways in which the 22 letters of the Hebrew alphabet can be arranged. He was interested in this calculation because the Jewish mystics believed that God had created the world and everything in it by naming these things (in Hebrew, of course): "God drew them, combined them, weighed them, interchanged them, and through them produced the whole creation and everything that is destined to be created. . . . Two stones [letters] build two houses [words], three build six houses, four build twenty-four houses, five build one hundred and twenty houses, six build seven hundred and twenty houses, seven build five thousand and forty houses."[10] Evidently, the author understood that the number of possible arrangements of n letters was $n!$. An Italian rabbi, Shabbetai Donnolo (913–970), derived this factorial rule very explicitly in a commentary on the *Sefer Yetsirah*:

> The first letter of a two-letter word can be interchanged twice, and for each initial letter of a three-letter word the other letters can be interchanged to form two two-letter words—for each of three times. And all the arrangements there are of three-letter words correspond to each one of the four letters that can be placed first in a four-letter word: a three-letter word can be formed in six ways, and so for every initial letter of a four-letter word there are six ways—altogether making twenty-four words, and so on.[11]

10.3.1 The Work of Abraham ibn Ezra

Although the author of the *Sefer Yetsirah* briefly mentioned how to calculate the number of combinations of letters taken two at a time, a more detailed study of combinations was carried out by Rabbi Abraham ben Meir ibn Ezra (1090–1167), a Spanish-Jewish philosopher, astrologer, and biblical commentator. It was in an astrological text that ibn Ezra discussed the number of possible conjunctions of the seven "planets" (including the sun and the moon). It was believed that these conjunctions would have a powerful influence on human life. Ibn Ezra thus calculated C_k^7 for each integer k from 2 to 7 and noted that the total was 120. He began with the simplest case, that the number of binary conjunctions was 21. This number was equal to the sum of the integers from one to six and could be calculated by ibn Ezra's rule for the sum of the integers from one up to a particular number: multiply that number by its half and by half of unity. In modern terms, we can write this as

$$C_2^n = \sum_{i=1}^{n-1} i = (n-1)\left(\frac{n-1}{2}\right) + (n-1)\left(\frac{1}{2}\right) = \frac{n(n-1)}{2}.$$

To calculate ternary combinations, ibn Ezra explained, "We begin by putting Saturn with Jupiter and with them one of the others. The number of the others is five; multiply 5 by its half and by half of unity. The result is 15. And these are the conjunctions of Jupiter."[12] Namely, there are five ternary combinations involving Jupiter and Saturn, four involving Jupiter and Mars, but not Saturn, and so on. Hence, there are $C_2^6 = 15 (= 5 \cdot \frac{5}{2} + 5 \cdot \frac{1}{2})$ ternary conjunctions involving Jupiter. Similarly, to find the ternary conjunctions involving Saturn but not Jupiter, ibn Ezra needed to calculate the number of choices of two planets from the remaining five: $C_2^5 = 10$. He then found the ternary conjunctions involving Mars, but neither Jupiter nor Saturn, and finally concluded with the result

$$C_3^7 = C_2^6 + C_2^5 + C_2^4 + C_2^3 + C_2^2 = 15 + 10 + 6 + 3 + 1 = 35.$$

Ibn Ezra next calculated the quaternary conjunctions by analogous methods. The conjunctions involving Jupiter require choosing three planets from the remaining six. Those with Saturn but not Jupiter require choosing three from five. So finally, $C_4^7 = C_3^6 + C_3^5 + C_3^4 + C_3^3 = 20 + 10 + 4 + 1 = 35$. Ibn Ezra then just stated the results for the conjunctions involving five, six, and seven planets. Essentially, he had given an argument for the case $n = 7$, easily generalizable to the general combinatorial rule:

$$C_k^n = \sum_{i=k-1}^{n-1} C_{k-1}^i.$$

In a later work, ibn Ezra essentially repeated the calculations above for C_2^7 and C_3^7 and then noted that by symmetry $C_4^7 = C_3^7$ and $C_5^7 = C_2^7$, something not explicitly mentioned by either ibn Mun'im or ibn al-Bannā in their own similar derivations somewhat later. Also, in a work on arithmetic in 1146, ibn Ezra introduced the Hebrew-speaking community to the decimal place value system. He used the first nine letters of the Hebrew alphabet to represent the first nine numbers and then instructed his readers on the meaning of place value, the use of the zero (which he wrote as a circle), and the various algorithms for calculation in the Hindu-Arabic system.

10.3.2 Levi ben Gerson and Induction

Early in the fourteenth century, Levi ben Gerson gave careful, rigorous proofs of various combinatorial formulas in a major work, the *Maasei Ḥoshev* (*The Art of the Calculator*) (1321, with a second edition in 1322). Levi's text is divided into two parts, a first theoretical part in which every theorem receives a detailed proof, and a second "applied" part in which explicit instructions are given for performing various types of calculation. (Levi used ibn Ezra's "Hebrew" place value system in this part.) Levi's theoretical first section began with a quite modern justification for considering theory at all:

> Because the true perfection of a practical occupation consists not only in knowing the actual performance of the occupation but also in its explanation, why the work is done in a particular way, and because the art of calculating is a practical occupation, it is clear that it is pertinent to concern oneself with its theory. There is also a second reason to inquire about the theory in this field. Namely, it is clear that this field contains many types of operations, and each type itself concerns so many different types of material that one could believe that they cannot all belong to the same subject. Therefore, it is only with the greatest difficulty that one can achieve understanding of the art of calculating, if one does not know the theory. With the knowledge of the theory, however, complete mastery is easy. One who knows it will understand how to apply it in the various cases which depend on the same foundation. If one is ignorant of the theory, one must learn each kind of calculation separately, even if two are really one and the same.[13]

Of course, as in any mathematical work, the reader must know the prerequisites, in this case Books VII, VIII, and IX of Euclid's *Elements*, "since it is not our intention in this book to repeat [Euclid's] words." But Levi did insist on giving careful, Euclidean-style proofs of all his results. The most important aspects of Levi's work are the combinatorial theorems. It is here that he used, somewhat more explicitly than his Islamic predecessors, the essentials of the method of mathematical induction, what he calls the process of "rising step by step without end." In general, when Levi used such a proof, he first proved the **inductive step**, the step that allows one to move from k to $k + 1$, next noted that the process begins at some small value of k, and then finally gave the complete result. Nowhere did he state the modern principle of induction, but it does appear that he knew how to use it. In fact, he used it initially in connection with two of the earliest theorems in the book, theorems that deal with associativity and commutativity of multiplication.

PROPOSITION 9 *If one multiplies a number which is the product of two numbers by a third number, the result is the same as when one multiplies the product of any two of these three numbers by the third.*

PROPOSITION 10 *If one multiplies a number which is the product of three numbers by a fourth number, the result is the same as when one multiplies the product of any three of these four numbers by the fourth.*

In modern notation, the first result states that $a(bc) = b(ac) = c(ab)$, while the second extends that result to four factors. The proof of Proposition 9 simply involves counting the number of times the various factors of the product appear in that product. In the proof of Proposition 10, Levi noted that $a(bcd)$ contains bcd a times. Since by Proposition 9, bcd can be thought of as $b(cd)$, it follows that the product $a(bcd)$ contains acd b times, or, $a(bcd) = b(acd)$, as desired. Levi then generalized these two results to any number of factors: "By the process of rising step by step without end, this is proved; that is, if one multiplies a number which is the product of four numbers by a fifth number, the result is the same as when one multiplies the product of any four of these by the other number. Therefore, the result of multiplying any product of numbers by another number contains any of these numbers as many times as the product of the others."[14] We see here the essence of the principle of mathematical induction. Levi used the principle again in proving that $(abc)d = (ab)(cd)$ and concluded that one can use the same proof to demonstrate the result without end: Any number contains the product of two of its factors as many times as the product of the remaining factors.

BIOGRAPHY

Levi ben Gerson (1288–1344)

Levi was born probably in the village of Bagnols-sur-Cèze in the south of France and spent most of his life in the nearby town of Orange. He was not only a mathematician but also an astronomer, philosopher, and biblical commentator. Not much is known of his life, except that he maintained contact with many important Christians, at the request of some of whom he composed a set of astronomical tables. His various works show that he was acquainted with the major Greek philosophical, astronomical, and mathematical writings, as well as with significant parts of the Islamic mathematical tradition. His best-known contribution to astronomy is his invention of the Jacob Staff (Fig. 10.10), which was used for centuries to measure the angular separation between heavenly bodies. In particular, it was popular with sixteenth-century European sailors who used it for navigation purposes.

FIGURE 10.10
Jacob Staff, invented by Levi ben Gerson

Levi was certainly not consistent about applying his induction principle. The middle of the text contains many theorems dealing with sums of various sequences of integers, theorems that could be proved by induction. But for many of these, Levi used other methods. For example, in proving that the sum of the first n integers equals $\frac{1}{2}n(n+1)$ (where n is even), he used the idea that the sums of the first and last integers, the second and next to last, and so on, are each equal to $n+1$. The same result when n is odd is proved by noting that those same sums are equal to twice the middle integer. In his proof of the formula for the sum of the first n integral cubes, however, he did use induction, in a way reminiscent of al-Karajī's proof of the same result. The basic inductive step is

PROPOSITION 41 *The square of the sum of the natural numbers from 1 up to a given number is equal to the cube of the given number added to the square of the sum of the natural numbers from 1 up to one less than the given number. [In modern notation, the theorem says that* $(1+2+\cdots+n)^2 = n^3 + (1+2+\cdots+(n-1))^2$*.]*

We present Levi's proof in modern notation. First, $n^3 = n \cdot n^2$. Also, $n^2 = (1+2+\cdots+n) + (1+2+\cdots+(n-1))$. (This result is Levi's Proposition 30.) Then

$$n^3 = n[(1+2+\cdots+n) + (1+2+\cdots+(n-1))]$$
$$= n^2 + n[2(1+2+\cdots+(n-1))].$$

But

$$(1+2+\cdots+n)^2 = n^2 + 2n(1+2+\cdots+(n-1)) + (1+2+\cdots(n-1))^2.$$

It follows that $n^3 + (1+2+\cdots+(n-1))^2 = (1+2+\cdots+n)^2$.

Levi next noted that although 1 has no number preceding it, "its third power is the square of the sum of the natural numbers up to it." In other words, he gave the first step of a proof by induction for the result stated as

PROPOSITION 42 *The square of the sum of the natural numbers from 1 up to a given number is equal to the sum of the cubes of the numbers from 1 up to the given number.*

Levi's proof is not quite what we would expect of a proof by induction. Instead of arguing from n to $n+1$, he argued, as did al-Karajī, from n to $n-1$. He noted that, first of all, $(1+2+\cdots+n)^2 = n^3 + (1+2+\cdots+(n-1))^2$. The final summand is, also by the previous proposition, equal to $(n-1)^3 + (1+2+\cdots+(n-2))^2$. Continuing in this way, Levi eventually reached $1^2 = 1^3$, and the result is proved. We note further that, although the proposition is stated in terms of an arbitrary natural number, in his proof Levi wrote only a sum of five numbers in his first step rather than the n used in our adaptation. The five are represented by the five initial letters of the Hebrew alphabet. Like many of his predecessors, Levi had no way of writing the sum of arbitrarily many integers and so used the method of generalizable example. Nevertheless, the idea of a proof by induction is evident in Levi's demonstration.

Inductive proofs are also evident in the final section of the theoretical part of the *Maasei Ḥoshev*, that on permutations and combinations. Levi's first result in this section showed that the number of permutations of a given number n of elements is what we call $n!$:

PROPOSITION 63 *If the number of permutations of a given number of different elements is equal to a given number, then the number of permutations of a set of different elements containing one more number equals the product of the former number of permutations and the given next number.*

Symbolically, the proposition states that $P_{n+1} = (n+1)P_n$ (where P_k stands for the number of permutations of a set of k elements). This result provides the inductive step in the proof of the proposition $P_n = n!$, although Levi did not mention that result until the end. His proof of proposition 63 was very detailed. Given a permutation, say, $abcde$, of the original n elements and a new element f, he noted that $fabcde$ is a permutation of the new set. Because there are P_n such permutations of the original set, there are also P_n permutations of the new set beginning with f. Also, if one of the original elements, for example, e, is replaced by the new element f, there are P_n permutations of the set a, b, c, d, f and therefore also P_n permutations of the new set with e in the first place. Because any of the n elements of the original set, as well as the new element, can be put in the first place, it follows that the number of permutations of the new set is $(n+1)P_n$. Levi finished the proof of Proposition 63 by showing that all of these $(n+1)P_n$ permutations are different. He then concluded, "Thus it is proved that the number of permutations of a given set of elements is equal to that number formed by multiplying together the natural numbers from 1 up to the number of given elements. For the number of permutations of 2 elements is 2, and that is equal to $1 \cdot 2$, the number of permutations of 3 elements is equal to the product $3 \cdot 2$, which is equal to $1 \cdot 2 \cdot 3$, and so one shows this result further without end."[15] Namely, Levi mentioned the beginning step and then noted that with the inductive step already proved, the complete result is also proved.

After proving, using a counting argument, that $P_2^n = n(n-1)$ (where P_k^n represents the number of permutations of k elements in a set of n), Levi proved that $P_k^n = n(n-1)(n-2)\cdots(n-k+1)$ by induction on k. As before, he stated the inductive step as a theorem:

PROPOSITION 65 *If a certain number of elements is given and the number of permutations of order a number different from and less than the given number of elements is a third number, then the number of permutations of order one more in this given set of elements is equal to* *the number which is the product of the third number and the difference between the first and the second numbers.*

Modern symbolism replaces Levi's convoluted wording with a brief phrase: $P^n_{j+1} = (n - j)P^n_j$. Levi's proof is quite similar to that of Proposition 63. At the end, he stated the complete result: "It has thus been proved that the permutations of a given order in a given number of elements are equal to that number formed by multiplying together the number of integers in their natural sequence equal to the given order and ending with the number of elements in the set."[16] To clarify this statement, Levi first gave the initial step of the induction by quoting his previous result in the case $n = 7$, that is, the number of permutations of order 2 in a set of 7 is equal to $6 \cdot 7$. Then the number of permutations of order 3 is equal to $5 \cdot 6 \cdot 7$ (since $5 = 7 - 2$). Similarly, the number of permutations of order 4 is equal to $4 \cdot 5 \cdot 6 \cdot 7$, "and so one proves this for any number."

In the final three propositions of the theoretical part of *Maasei Ḥoshev*, Levi completed his development of formulas for permutations and combinations. Proposition 66 showed that $P^n_k = C^n_k P^k_k$, while Proposition 67 simply rewrote this as $C^n_k = P^n_k / P^k_k$. Since he had already given formulas for both the numerator and denominator of this quotient, Levi thus had demonstrated the standard formula for C^n_k:

$$C^n_k = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}.$$

And finally, Proposition 68 demonstrated that $C^n_k = C^n_{n-k}$.

Levi gave examples of many of these results in the second section of his book. For example, he noted that to determine the sum of the cubes of the numbers from 1 to 6, one first calculates that the sum of the numbers themselves is 21 and therefore the sum of the cubes is the square of 21, namely, 441. Or to find the number of permutations of five elements out of a set of eight, P^8_5, one multiplies $4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$ to get 6720. Then the number of combinations of five elements out of eight, namely, C^8_5, is that number divided by $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, or 120. The result is 56.

Finally, at the end of the second section, Levi presented a number of interesting problems, most seemingly "practical," and most of which could be solved through a knowledge of ratio and proportion. These problems, including some familiar ones, become progressively harder, but Levi gave a detailed explanation of the solution to each (Sidebar 10.2). Two of these problems are included in the exercises to this chapter.

10.4 MEDIEVAL ALGEBRA

Although the theory of combinatorics appears to have developed in Europe through the Jewish tradition, the writers on algebra in medieval Europe were direct heirs to Islamic work.

10.4.1 Leonardo of Pisa's *Liber Abbaci*

One of the earliest European writers on algebra was Leonardo of Pisa, most famous for his masterpiece, the *Liber abbaci*, or *Book of Calculation*. (The word *abbaci,* from *abacus,* does not refer to a computing device but simply to calculation in general.) The first edition

SIDEBAR 10.2 *Did Anyone Read the Works of Levi ben Gerson?*

Although the *Maasei Ḥoshev* was the earliest work in Europe to consider the combinatorial formulas in detail as well as to provide examples of proof by mathematical induction, it does not seem to have had any influence in the subsequent centuries. As far as can be determined, there are no references to this work in any later European mathematical work, and in fact the combinatorial formulas themselves do not appear anywhere in Europe for the next 200 years, nor is proof by induction used before the work of Pascal in the mid-seventeenth century. So what happened to Levi's book? Did anyone read it?

The simple answer to the last question is, yes. There are today about a dozen manuscript copies of the work extant, in libraries throughout Europe as well as one copy in New York, most written in the fifteenth and sixteenth centuries. For a medieval manuscript, that is not a trivial number of copies. And for some copies, we know the name of the copyist or the original owner. In fact, the copy in London was at one time owned by Mordecai Finzi of Mantua, the fifteenth-century Jewish scientist who translated the work of Abu Kāmil into Hebrew. The more important question then is, Did anyone read the *Maasei Ḥoshev* who used it to continue work in the field of combinatorics?

As far as Finzi is concerned, there is no record of his ever having written about combinatorics. On the other hand, we do know that Marin Mersenne wrote about combinatorics in his works on music theory in the mid-1630s. And his methodology bears some resemblance to the work of Levi. Could he have read Levi's work or learned of it through one of his many correspondents? For that to have happened, there would have had to be a copy of the manuscript in Paris available for someone who both read Hebrew and understood mathematics. In fact, all of these conditions are satisfied. A copy of the *Maasei Ḥoshev* was brought to Paris around 1620 by Achille Harlay de Sancy, the French ambassador to Constantinople.

De Sancy donated the manuscript—as part of his large collection of Hebrew manuscripts—to the library of the Oratorian priests, whose Paris house he joined as well. There the manuscript remained until the Oratorian houses were closed in the 1790s during the French Revolution. Now Mersenne was certainly in contact with many priests at the Oratory, and we also know that some of them read Hebrew and were trained in mathematics. But there the trail ends for now. The manuscript itself has no notes on it, nor are there library records from the Oratory that tell us who may have looked at the manuscript. So we may never know the answer to the question posed above.

of this work appeared in 1202, while a slightly revised one was published in 1228. The many surviving manuscripts testify to the wide readership the book enjoyed. The sources for the *Liber abbaci* were largely in the Islamic world, which Leonardo visited during many journeys, but he enlarged and arranged the material he collected through his own genius. The book contained not only the rules for computing with the new Hindu-Arabic numerals, but also numerous problems of various sorts in such practical topics as calculation of profits, currency conversions, and measurement, supplemented by the now standard topics of current algebra texts such as mixture problems, motion problems, container problems, the Chinese remainder problem, and, at the end, various forms of problems solvable by use of quadratic equations. Interspersed among the problems is a limited amount of theory, such as methods for summing series, geometric justifications of the quadratic formulas, and even a brief discussion of negative numbers.

Leonardo used a great variety of methods in his solution of problems. Often, in fact, he used special procedures designed to fit a particular problem rather than more general methods. One basic method used often is the old Egyptian method of "false position" in which a convenient, but wrong, answer is given first and then adjusted appropriately to get the correct result. Similarly, he used the method of "double false position," a method that has its origins in China but was also used in medieval Islam. Leonardo also used the methods of al-Khwārizmī for solving quadratic equations. For many of the problems, it is possible to cite Leonardo's sources. He often took problems verbatim from such Islamic mathematicians as al-Khwārizmī, Abū Kāmil, and al-Karajī, many of which he found in Arabic manuscripts discovered in his travels. Some of the problems seem ultimately to have come from China or India, but Leonardo probably learned these in Arabic translations. The majority of the problems, however, are of his own devising and show his creative abilities. A few of Leonardo's problems and solutions should give the flavor of this most influential mathematical work.

Leonardo began his text by introducing the Hindu-Arabic numerals: "The nine Indian figures are 9, 8, 7, 6, 5, 4, 3, 2, 1. With these nine figures, and with the sign 0, which the Arabs call *zephir* (cipher), any number whatsoever is written, as is demonstrated below."[17] He then showed precisely that, giving the names to the various places in the place value system (for integers only). Leonardo next dealt with various algorithms for adding, subtracting, multiplying, and dividing whole numbers and common fractions. His notation for mixed numbers differed from ours in that he wrote the fractional part first, but his algorithms are generally close to the ones we use today. For example, to divide 83 by $5\frac{2}{3}$ (or, as he writes, $2/3\ 5$), Leonardo multiplied 5 by 3 and added 2, giving 17. He then multiplied 83 by 3, giving 249, and finally divided 249 by 17, giving $14\frac{11}{17}$. To add $1/5 + 3/4$ to $1/10 + 2/9$, Leonardo multiplied the first two denominators, 4 and 5, to get 20, then multiplied this by the denominator 9 to get 180. A multiplication by 10 was unnecessary since 10 is already a factor of 180. Then $1/5 + 3/4$ times 180 is 171, while $1/10 + 2/9$ times 180 is 58. The sum of these two, 229, is then divided by 180 to get the final result, $1\frac{49}{180}$. Leonardo wrote the answer as $\frac{1\ 6\ 2}{2\ 9\ 10}1$, by which he meant $1 + \frac{1}{2\cdot 9\cdot 10} + \frac{6}{9\cdot 10} + \frac{2}{10}$. This latter notation perhaps derives from the Pisan monetary system. Because 1 pound is divided into 20 *solidi* and each *solidus* is divided into 12 *denarii*, it was convenient for him, for example, to write 17 pounds, 11 *solidi*, 5 *denarii* as $\frac{5\ 11}{12\ 20}17$. Notations aside, Leonardo was able to use his procedures effectively to show his readers how to perform the intricate calculations needed to convert among the many currencies in use in the Mediterranean basin during his time.

Leonardo presented several versions of the classic problem of buying birds. In the first, he asked how to buy 30 birds for 30 coins, if partridges cost 3 coins each, pigeons 2 coins each, and sparrows 2 for 1 coin. He began by noting that he could buy 5 birds for 5 coins by taking 4 sparrows and 1 partridge. Similarly, 2 sparrows and 1 pigeon give him 3 birds for 3 coins. By multiplying the first transaction by 3 and the second by 5, he procured 12 sparrows and 3 partridges for 15 coins and 10 sparrows and 5 pigeons also for 15 coins. Adding these two transactions gave the desired answer: 22 sparrows, 5 pigeons, 3 partridges.

Another classic problem is that of the lion in the pit: The pit is 50 feet deep. The lion climbs up 1/7 of a foot each day and then falls back 1/9 of a foot each night. How long will it take him to climb out of the pit? Leonardo here used a version of "false position." He assumed the answer to be 63 days, since 63 is divisible by both 7 and 9. Thus, in 63 days the lion will climb up 9 feet and fall down 7, for a net gain of 2 feet. By proportionality, then, to climb 50 feet, the lion will take 1575 days. (By the way, Leonardo's answer is incorrect. At the end of 1571 days, the lion will be 8/63 of a foot from the top. On the next day, he will reach the top.)

Leonardo's example of the Chinese remainder problem asked to find a number that when divided by 2 had remainder 1, by 3 had remainder 2, by 4 had remainder 3, by 5 had remainder 4, by 6 had remainder 5, and by 7 had remainder 0. To solve this, he noted that 60 was evenly divisible by 2, 3, 4, 5, and 6. Therefore, $60 - 1 = 59$ satisfied the first five conditions as did any multiple of 60, less 1. Thus, he had to find a multiple of 60 that had remainder 1 on division by 7. The smallest such number is 120, and therefore 119 is the number sought. (Interestingly, this problem was also posed by ibn al-Haytham two centuries earlier.)

Negative numbers appear in one of Leonardo's many problems dealing with a purse found by a number of men. In this particular problem, there are 5 men. The amount the first has together with the amount in the purse is $2\frac{1}{2}$ times the total of the amounts held by the other four. Similarly, the second man's amount together with the amount in the purse is $3\frac{1}{3}$ times the total held by the others. Analogously, the fraction is $4\frac{1}{4}$ for the third man, $5\frac{1}{5}$ for the fourth man, and $6\frac{1}{6}$ for the fifth man. Leonardo worked out the problem and discovered that the only way this can be solved is for the first man to begin with a debt of 49,154. In a few other problems, he also gave negative answers, and even demonstrated an understanding of the basic rules for adding and subtracting with these numbers.

Leonardo used many methods to solve his problems, but in later chapters of the book he tended toward methods that are explicitly algebraic. In fact, Leonardo credited the Arabs with what he called the "direct" method of solution, a method that involves setting up an equation and then simplifying it according to standard rules. For example, suppose two men have some money, and one said to the other: If you will give me 7 of your *denarii*, then I will have five times as much as you. The other said, if you give me 5 *denarii*, then I will seven times as much as you. Leonardo started by assuming that the second man has "thing" plus 7 *denarii*. Then the first man has five things minus 7. If the first then gives 5 to the second, he will have five things minus 12, while the second man will have thing plus 12. The equation is then "one thing and 12 *denarii* are seven times five things minus 12 *denarii*." Leonardo then solved the equation to find that "thing" is $2\frac{14}{17}$ *denarii*, and therefore that the second man began with $9\frac{14}{17}$ *denarii*, while the first began with $7\frac{2}{17}$ *denarii*.

Leonardo also dealt comfortably with determinate and indeterminate problems in more than two unknowns. For example, suppose there are four men such that the first, second, and third together have 27 *denarii*, the second, third, and fourth together have 31, the third, fourth, and first have 34, while the fourth, first, and second have 37. To determine how much each man has requires solving a system of four equations in four unknowns. Leonardo accomplished this expeditiously by adding the four equations together to determine that four times the total sum of money equals 129 *denarii*. The individual amounts are then easily calculated. On the other hand, in a similar question reducible to the four equations $x + y = 27$, $y + z = 31$, $z + w = 34$, $x + w = 37$, Leonardo first noted that this system is impossible since the two different ways of calculating the total sum of money give two different answers, 61 and 68. However, if one changes the fourth equation to $x + w = 30$, one can simply choose x arbitrarily ($x \leq 27$) and calculate y, z, and w by using the first, second, and third equations, respectively.

The most famous problem of the *Liber abbaci*, the rabbit problem, is tucked inconspicuously between a problem on perfect numbers and the problem just discussed: "How many pairs of rabbits are created by one pair in one year? A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also."[18] Leonardo proceeded to calculate: After the first month there will be two pairs, after the second, three. In the third month, two pairs will produce, so at the end of that month there will be five pairs. In the fourth month, three pairs will produce, so there will be eight. Continuing in this fashion, he showed that there will be 377 pairs by the end of the twelfth month. Listing the sequence 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 in the margin, he noted that each number is found by adding the two previous numbers, and "thus you can do it in order for an infinite number of months." This sequence, calculated recursively, is known today as a **Fibonacci sequence**. It turns out that it has many interesting properties unsuspected by Leonardo, not the least of which is its connection with the Greek problem of dividing a line in extreme and mean ratio.

In his final chapter, Leonardo demonstrated his complete command of the algebra of his Islamic predecessors as he showed how to solve equations that reduce ultimately to quadratic equations. He discussed in turn each of the six basic types of quadratic equations, as given by al-Khwārizmī, and then gave geometric proofs of the solution procedures for each of the three mixed cases. He followed the proofs with some 50 pages of examples, most taken from the works of al-Khwārizmī and Abū Kāmil, including the familiar ones beginning with "divide 10 into two parts." In particular, he included the latter's problem of three equations in three unknowns, discussed in Chapter 9.

The content of the *Liber abbaci* contained no particular advance over mathematical works then current in the Islamic world. In fact, as far as the algebra was concerned, Leonardo was only presenting tenth-century Islamic mathematics and ignoring the advances of the eleventh and twelfth centuries. The chief value of the work, nevertheless, was that it did provide Europe's first comprehensive introduction to Islamic mathematics. Those reading it were afforded a wide variety of methods to solve mathematical problems, methods that provided the starting point from which further progress could ultimately be made.

10.4.2 The *Liber Quadratorum*

Another briefer work by Leonardo, the *Liber quadratorum* (*Book of Squares*) of 1225, is much more theoretical than the *Liber abbaci*. This is a book on number theory, in which Leonardo discussed the solving in rational numbers of various equations involving squares. The book originated in a question posed to Leonardo by a Master John of Palermo, a member of the entourage of the Holy Roman Emperor Frederick II, whom Leonardo met as described in the opening of this chapter. According to Leonardo, Master John proposed the question, to "find a square number from which, when five is added or subtracted, always arises a square number. Beyond this question, the solution of which I have already found, I saw, upon reflection, that this solution itself and many others have origin in the squares and the numbers which fall between the squares."[19] The initial problem, to find x, y, z, so that $x^2 + 5 = y^2$ and $x^2 - 5 = z^2$, is solved as the seventeenth of the 24 propositions of the book, but Leonardo first developed various properties of square numbers and sums of square numbers. Interestingly, John of Palermo was not only a mathematician but also an Arabic scholar, who may well have been familiar with this problem from the work of al-Karajī.

To solve Master John's problem, Leonardo introduced what he called **congruous** numbers, numbers n of the form $ab(a + b)(a - b)$ when $a + b$ is even and $4ab(a + b)(a - b)$ when

NONBIOGRAPHY

Jordanus de Nemore

Although Jordanus has been recognized as one of the best mathematicians of the Middle Ages, there is virtually no available evidence about his life, other than that he appears to have been connected with the University of Paris in the early decades of the thirteenth century. Some years ago, he was identified with Jordanus de Saxonia, the second Master General of the Dominican order, but recent scholarly work has shown that this identification is impossible. The translator of *De numeris datis*, Barnabas Hughes, concludes that Jordanus is *sine patre, sine matre, sine genealogia*. He also notes in a letter, however, that "the only explanation that appealed to me [as to why no biographical information is extant] was that the name is a pseudonym. But why a *nom de plume*? Could it be that Jordanus was really a woman? Shades of Hypatia! Thirteenth century women were good for writing poems, songs and prayers; but science?"[21]

$a+b$ is odd. He showed that congruous numbers are always divisible by 24 and that integral solutions of $x^2+n=y^2$ and $x^2-n=z^2$ can be found only if n is congruous. The original problem is therefore not solvable in integers. Nevertheless, since $720=12^2\cdot 5$ is a congruous number (with $a=5$ and $b=4$) and since $41^2+720=49^2$ and $41^2-720=31^2$, it follows by dividing both equations by 12^2 that $x=41/12$, $y=49/12$, $z=31/12$, provides a solution in rational numbers to $x^2+5=y^2$, $x^2-5=z^2$. Leonardo's methodology is different from that of al-Karajī, although he does get the same answer, but it is similar to that in other Islamic treatises on number theory of the same time period, including the work of Abū Ja-far al-Khāzin (tenth century).[20]

It is clear that Leonardo mastered the Islamic mathematics he had learned in his travels and passed what he knew on to his European successors. With respect to the number theory of the *Liber Quadratorum*, Leonardo had no successor until several centuries later when Diophantus's *Arithmetica* was again available in Europe. On the other hand, the practical material in the *Liber abbaci* and the *Practica geometriae* was picked up by Italian surveyors and masters of computation (*maestri d'abbaco*), who were influential in the next several centuries in bringing a renewed sense of mathematics into Italy. It took a full 300 years, however, for this renewed mathematical knowledge to increase to the point where conditions in Italy were advanced enough for new mathematics to be created.

10.4.3 **Jordanus de Nemore**

One of the first mathematicians to make some advance over the work of Leonardo was a contemporary, Jordanus de Nemore. Although we know virtually nothing about the author himself, it is believed that he taught in Paris around 1220. His writings include several works on arithmetic, geometry, astronomy, mechanics, and algebra, and it appears that he worked to create a Latin version of the quadrivium, based upon a theoretical work on arithmetic. Jordanus's *Arithmetica* was far different from the demonstrationless arithmetic of Boethius, which was then circulating widely in Europe. Jordanus's work was firmly based on a Euclidean model, with definitions, axioms, postulates, propositions, and careful proofs. Also like Euclid, Jordanus did not give any numerical examples.

The *Arithmetica*, a work in 10 books, dealt with such topics found in Euclid as ratio and proportion, prime and composite numbers, the Euclidean algorithm, and the geometrical algebra propositions of *Elements*, Book II. It also considered much material not found in Euclid, including figurate numbers and a detailed study of named ratios, due to Nicomachus. Most interesting, however, are a few items not found in the Greek sources. For example, in Book VI Jordanus solved a problem virtually identical to the central problem of Leonardo's *Book of Squares*.

PROPOSITION VI–12 *To find three square numbers whose continued differences are equal.*

In modern symbols, Jordanus wanted to determine y^2, x^2, and z^2 so that $y^2 - x^2 = x^2 - z^2$. His solution amounted to setting

$$y = \frac{a^2}{2} + ab - \frac{b^2}{2}, \qquad x = \frac{a^2 + b^2}{2}, \qquad z = \frac{b^2}{2} + ab - \frac{a^2}{2},$$

where a, b, have the same parity. In contrast to Leonardo, Jordanus was just interested in integral solutions, but did not give any example. Nevertheless, it is straightforward to see that the difference of the squares in Jordanus's theorem is a congruous number according to Leonardo's definition.

And in Book IX, Jordanus displayed the "Pascal" triangle, for the first time in a European work. The construction of the triangle is the standard one:

> Put 1 at the top and below two 1's. Then the row of two 1's is doubled so that the first 1 is in the first place and another 1 in the last place as in the second row; and 1,2,1 will be in the third row. The numbers are added two at a time, the first 1 to the 2 in the second place, and so on through the row until a final 1 is put at the end. Thus the fourth row has 1,3,3,1. In this way subsequent numbers are made from pairs of preceding numbers.[22]

Most of the medieval manuscripts of the *Arithmetica* display a version of the triangle at this point, some even up to the tenth line. Jordanus then used the triangle explicitly in Proposition IX–70 to construct series of terms in given ratios. For example, if $a = b = c = d = 1$, then the numbers $e = 1a = 1$, $f = 1a + 1b = 2$, $g = 1a + 2b + 1c = 4$, and $h = 1a + 3b + 3c + 1d = 8$ form a continued proportion with constant ratio 2. Similarly, $k = 1e = 1$, $\ell = 1e + 1f = 3$, $m = 1e + 2f + 1g = 9$, and $n = 1e + 3f + 3g + 1h = 27$ form a continued proportion with ratio 3.

Jordanus's *Arithmetica* was widely read, at least judging from the number of extant manuscripts. Similarly, his major work on algebra, *De numeris datis* (*On Given Numbers*), also had a large circulation in medieval Europe. *De numeris datis* is an analytic work on algebra, based on but differing in spirit from the Islamic algebras that had made their way into Europe by the early thirteenth century. It appears to be modeled on Euclid's *Data*, available to Jordanus in a Latin translation of Gerard of Cremona, in that it presents problems in which certain quantities are given and then shows that other quantities are therefore also determined. The problems in *De numeris datis*, however, are algebraic rather than geometric. Jordanus's proofs are also algebraic, or, perhaps, arithmetical. In fact, one of his aims is apparently to base the new algebra on arithmetic, the most fundamental of the subjects of the quadrivium, rather than on geometry, and especially on his own work on the subject. He also organized his book in a logical fashion and, in a major departure from his Euclidean model, and even from his own *Arithmetica*, provided numerical examples for most of his theoretical results.

Although many of the actual problems and the numerical examples were available in the Islamic algebras, Jordanus adapted them to his own purposes. In particular, he made the major change of using letters to stand for arbitrary numbers. Jordanus's algebra was no longer entirely rhetorical. That is not to say that his symbolism was modern-looking. He picked his letters in alphabetical order with no distinction between letters representing known quantities and those representing unknowns and used no symbols for operations. Sometimes a single number was represented by two letters. At other times the pair of letters ab represented the sum of the two numbers a and b. The basic arithmetic operations were always written in words. And Jordanus did not use the new Hindu-Arabic numerals. All of his numbers were written as Roman numerals. Nevertheless, the idea of symbolism, so crucial to any major advance in algebraic technique, was found, at least in embryonic form, in Jordanus's work.

To understand Jordanus's contribution, we consider a few of the text's more than 100 propositions, organized into four books. Like Euclid, Jordanus wrote each proposition in a standard form. The general enunciation was followed by a restatement in terms of letters. By use of general rules, the letters representing numbers were manipulated into a canonical form from which the general solution could easily be found. Finally, a numerical example was calculated following the general outlines of the abstract solution. The canonical forms themselves are among the earliest of the propositions.

PROPOSITION I–1 *If a given number is divided into two parts whose difference is given, then each of the parts is determined.*

Jordanus's proof was straightforward: "Namely, the lesser part and the difference make the greater. Thus the lesser part with itself and the difference make the whole. Subtract therefore the difference from the whole and there will remain double the lesser given number. When divided [by two], the lesser part will be determined; and therefore also the greater part. For example, let 10 be divided in two parts of which the difference is 2. When this is subtracted from 10 there remains 8, whose half is 4, which is thus the lesser part. The other is 6."[23]

In modern symbolism, Jordanus's problem amounted to the solution of the two equations $x + y = a$, $x - y = b$. Jordanus noted first that $y + b = x$, so that $2y + b = a$ and therefore $2y = a - b$. Thus, $y = \frac{1}{2}(a - b)$ and $x = a - y$.

Jordanus used this initial proposition in many of the remaining problems of Book I. For example, consider

PROPOSITION I–3 *If a given number is divided into two parts, and the product of one by the other is given, then of necessity each of the two parts is determined.*

This proposition presented one of the standard Babylonian problems: $x + y = m$, $xy = n$. Jordanus's method of solution, however, is different from the classic Babylonian solution, and, in addition, he used symbolism as indicated: Suppose the given number abc is divided into the parts ab and c. Suppose ab multiplied by c is d and abc multiplied by itself is e. Let f be the quadruple of d, and g be the difference of e and f. Then g is the square of the difference between ab and c. Its square root b is then the difference between ab and c. Since now both the sum and difference of ab and c are given, both ab and c are determined according to the first proposition. Jordanus's numerical example used 10 as the sum of the two parts and 21 as the product. He noted that 84 is quadruple 21, that 100 is 10 squared, and that 16 is their difference. Then the square root of 16, namely, 4, is the difference of the two parts of 10. By the proof of the first proposition, 4 is subtracted from 10 to get 6. Then 3 is the desired smaller part while 7 is the larger.

Jordanus's solution, translated into modern symbolism, used the identity $(x - y)^2 = (x + y)^2 - 4xy = m^2 - 4n$ to determine $x - y$ and reduce the problem to Proposition I–1. The solution is then $y = \frac{1}{2}(m - \sqrt{m^2 - 4n})$, $x = m - y$. Jordanus's method appears to be new with him, and he continued to use his own methods throughout the work. Even his solution of problems in Book I equivalent to pure quadratic equations used methods different from the standard ones used by the Islamic algebraists. Nevertheless, the numerical examples themselves have a familiar look. In fact, every proposition in Book I deals with a number divided into two parts, and in every example but one the number to be divided is 10. The solution methods may differ somewhat from those in the Islamic texts, but it is clear that al-Khwārizmī's problems live on!

Many of the propositions in the remaining three books of Jordanus's treatise dealt with numbers in given proportion. They demonstrated his fluency in dealing with the rules of proportion found in Books V and VII of Euclid's *Elements*, most of which are also found in his own *Arithmetica*. Consider

PROPOSITION II–18 *If a given number is divided into however many parts, whose continued proportions are given, then each of the parts is determined.*

Because Jordanus, like his contemporaries, had no way to express arbitrarily many "parts," he dealt in his proof with a number divided into three parts: $a = x + y + z$. Then $x : y = b$ and $y : z = c$ are both known ratios. Jordanus noted that the ratio $x : z$ is also known. It follows that the ratio of x to $y + z$ is known and therefore also that of a to x. Since a is known, x and then y and z can be determined. His example enables us to follow his verbal description. The number 60 is divided into three parts, of which the first is double the second and the second is triple the third. That is, $x + y + z = 60$, $x = 2y$, $y = 3z$. Then $x = 6z$ and therefore $y + z = \frac{2}{3}x$. So $60 = \left(1\frac{2}{3}\right)x$, and $x = 36$, $y = 18$, $z = 6$. One notes that Jordanus easily inverted ratios if necessary and also knew how to combine them.

Among the propositions in Book IV are three giving the three standard forms of the quadratic equation, all presented with algebraic rather than geometric justifications. For these problems, however, Jordanus did use the standard Islamic algorithms, but with his own symbolism. Consider

PROPOSITION IV–9 *If the square of a number added to a given number is equal to the number produced by multiplying the root and another given number, then two values are possible.*

Thus, Jordanus asserted that there are two solutions to the equation $x^2 + c = bx$. He then gave the procedure for solving the equation: Take half of b, square it to get f, and let g be the difference of x and $\frac{1}{2}b$, that is, $g = \pm(x - \frac{1}{2}b)$. Then $x^2 + f = x^2 + c + g^2$ and $f = c + g^2$. Jordanus concluded by noting that x may be obtained by either subtracting g from $b/2$ or by adding g to $b/2$, that is,

$$x = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}.$$

His example made his symbolic procedure clearer. To solve $x^2 + 8 = 6x$, he squared half of 6, giving 9, and then subtracted 8 from it, leaving 1. The square root of 1 is 1, and this is the difference between x and 3. Hence, x can be either 2 or 4.

Among the other quadratic problems Jordanus solved in Book IV are the systems $xy = a$, $x^2 + y^2 = b$, and $xy = a$, $x^2 - y^2 = b$. In each case, as in all the previous cases, the given example resulted in a positive integral answer. While Jordanus often used fractions as part of his solution, he carefully arranged matters so that final answers were always whole numbers. If, in fact, he had read Abū Kāmil's *Algebra*, which was available in Latin, Jordanus would have seen nonintegral, and even nonrational, solutions to this type of problem. He nevertheless rejected such solutions when he made up his examples. Given his very formal style, however, Jordanus may still have been under the influence of Euclid and may have felt that irrational "numbers" simply did not belong in a work based on arithmetic. Hence, although *De numeris datis* represented an advance from the Islamic works in the use of analysis, in the constant striving toward generality, and in some symbolization, it returned to the strict Greek separation of number from magnitude, an idea from which Jordanus's Islamic predecessors had already departed. Thus, it appears that although Jordanus certainly made use of the new Islamic material available in Europe, his goal was to provide his readers with a mathematics based as much as possible on Greek principles.

10.5 THE MATHEMATICS OF KINEMATICS

The algebraic work of Jordanus de Nemore was not developed further in the thirteenth century, even though a group of followers had appeared in Paris by the middle of that century. Perhaps Europe was not then ready to resume the study of pure mathematics. By early in the fourteenth century, however, certain other aspects of mathematics began to develop in the universities of Oxford and Paris out of attempts to clarify certain remarks in Aristotle's physical treatises (Sidebar 10.3).

10.5.1 The Study of Ratios

One of the new mathematical ideas came from the effort to derive a relationship among the force F applied to an object, its resistance R, and its velocity V. A basic postulate of medieval physics was that F must be greater than R for motion to be produced. (The medieval philosophers did not attempt to measure these quantities in any particular units.) The simplest relationship among these quantities implied by Aristotle's own words may be expressed by the statement that F/R is proportional to V. This mathematical relationship, however, quickly leads to a contradiction of the postulate. For if F is left fixed, the continual doubling of R is equivalent to the continual halving of V. Halving a positive velocity keeps it positive, but the doubling of R eventually makes R greater than F, thus contradicting the notion that F must be greater than R for motion to take place.

Thomas Bradwardine (1295–1349) of Merton College, Oxford, in his 1328 *Tractatus de proportionibus velocitatum in motibus* (*Treatise on the Proportions of Velocities in Movements*), proposed a solution to this dilemma, that is, a "correct" interpretation of Aristotle's

SIDEBAR 10.3 *The Medieval Universities*

It was during the late twelfth century that Europe saw the beginnings of the institutions that were to have immense influence in the development of science in general and mathematics in particular: the universities. We cannot assign any specific date for the origins of the earliest universities. They were formed as societies, or guilds, of masters and pupils and appeared on the scene when there was enough learning available in western Europe to justify their existence. The earliest of these institutions were in Paris, Oxford, and Bologna. In Paris, the university grew out of the cathedral school of Notre Dame. The masters and students gradually grouped themselves into the four faculties of arts, theology, law, and medicine. Although there is evidence of the existence of the university in the late twelfth century, the first official charter dates from 1200. The University at Oxford emerged not out of a church school but from a group of English students who had returned from Paris. Again, although the university certainly existed in the late twelfth century, the first official document dates from 1214. At Bologna, the university began as a law school, perhaps as early as the eleventh century. The Italian university differed from its northern counterparts, however, since it was a guild of students, rather than one of professors, that initially constituted the organization. The students elected the professors and other officials. Student control was somewhat weakened, however, because salaries were paid by the Bolognese municipality and the faculty conducted the examinations.

The curriculum in arts at all of the universities was based on the ancient trivium of logic, grammar, and rhetoric and the quadrivium of arithmetic, geometry, music, and astronomy (Fig. 10.11). This study in the faculty of arts provided the student with preparation for the higher faculties of law, medicine, or theology. The centerpiece of the arts curriculum was the study of logic, and the primary texts for this were the logical works of Aristotle, all of which had by this period been translated into Latin. The masters felt that logic was the appropriate first area of study since it taught the methods for all philosophic and scientific inquiry. Gradually, other works of Aristotle were also added to the curriculum. For several centuries, the great philosopher's works were the prime focus of the entire arts curriculum. Other authors were studied insofar as they allowed one to better understand this most prolific of the Greek philosophers. In particular, mathematics was studied only as it related to the work of Aristotle in logic or the physical sciences. The mathematical curriculum itself—the quadrivium—usually consisted of arithmetic, taken from such works as Boethius's adaptation of Nicomachus or a medieval text on rules for calculation; geometry, taken from Euclid and one of the practical geometries; music, taken also from a work of Boethius; and astronomy, taken from Ptolemy's *Almagest* and some more recent Latin translations of Islamic astronomical works.

FIGURE 10.11
The quadrivium—arithmetic, geometry, music, and astronomy—on two stamps from the Netherlands Antilles

remarks. The rule noted above implies that for two forces F_1, F_2, two resistances R_1, R_2, and two velocities V_1, V_2, the equation

$$\frac{F_2}{R_2} = \frac{V_2}{V_1}\frac{F_1}{R_1}$$

is satisfied. Bradwardine suggested that this should be replaced by the relationship expressed in modern notation as

$$\frac{F_2}{R_2} = \left(\frac{F_1}{R_1}\right)^{\frac{V_2}{V_1}}.$$

In other words, the multiplicative relation should be replaced by an exponential one. This solution indeed removed the absurdity noted above. Given initially that $F > R$ (or $F/R > 1$), halving the velocity in this situation is equivalent to taking roots of the ratio $\frac{F}{R}$. Consequently, F/R will remain greater than 1, and R will never be greater than F. Neither Bradwardine nor anyone else in this period, however, attempted to give any experimental justification for this relationship. The scholars at Merton wanted a mathematical explanation of the world, not a physical one. As it turned out, Bradwardine's idea was discarded as a physical principle by the middle of the next century, but the mathematics behind it led to important new ideas. To deal with these required a systematic study of ratios, in particular, of the idea of compounding (or multiplying) ratios.

Until the fourteenth century, compounding was performed in the classical Greek style. Thus, to deal with the ratio compounded of $a : b$ and $c : d$, one needed to find a magnitude e such that $c : d = b : e$. Then the desired compound ratio would be $a : e$. Gradually, however, the more explicit notion of multiplication of ratios was introduced. For example, Bradwardine's contemporary at Oxford, Richard of Wallingford, defined ratios as well as their compounding and dividing in part II of his *Quadripartitum*:

1. A **ratio** is a mutual relation between two quantities of the same kind.
2. When one of two quantities of the same kind divides the other, what results from the division is called the **denomination** of the ratio of the dividend to the divisor.
3. A ratio [is said to be] **compounded** of ratios when the product of the denominations gives rise to some denomination.
4. A ratio [is said to be] **divided** by a ratio when the quotient of the denominations gives rise to some denomination.[24]

There are several important notions here. First, Richard emphasized that ratios can be taken only between quantities of the same kind. This Euclidean idea meant that velocity could not be treated as a ratio of distance to time. Second, the word *denomination* in these definitions referred to the "name" of the ratio in "lowest terms," as given in the terminology due to Nicomachus now standard in Europe. For example, the ratio 3 : 1 was called a triple ratio, while that of 3 : 2 was called the sesquialter. Finally, definitions 3 and 4 showed that for Richard, unlike for Euclid, multiplication (of numbers) was involved in compounding, and the inverse notion of division could also be applied. Thus, although he compounded the ratios 4 : 16, 16 : 2, and 2 : 12 to get 4 : 12, he noted that since the first ratio is a subquadruple (1 : 4), the second an octuple (8 : 1), and the third a subsextuple (1 : 6), they can be compounded by first dividing 8 by 4 to get 2, and then dividing 2 by 6 to get 1 : 3 (a subtriple) as the final result. Thus, one can actually use the standard algorithm for multiplying fractions to "compound" ratios.

Nicole Oresme (1320–1382), a French cleric and mathematician associated with the University of Paris, undertook a very detailed study of ratios in his *Algorismus proportionum* (*Algorithm of Ratios*) and his *De proportionibus proportionum* (*On the Ratios of Ratios*). In addition to performing compounding in the traditional manner, Oresme noted explicitly that one can also compound ratios by multiplying the antecedents and then multiplying the consequents. Thus, 4 : 3 compounded with 5 : 1 is 20 : 3. The connecting link between the two methods is presumably that $a : b$ can be expressed as $ac : bc$, $c : d$ as $bc : bd$, and hence the compound of $a : b$ with $c : d$ as the compound of $ac : bc$ with $bc : bd$ or as $ac : bd$. In any case, given a way of multiplying two ratios, Oresme also noted that one could reverse the procedure and divide two ratios. Thus, the quotient of $a : b$ by $c : d$ was the ratio $ad : bc$.

Now that the product of any two ratios had been defined, Oresme discussed the product of a given ratio with itself. Thus, $a : b$ compounded with itself n times gives what would now be written as $(a : b)^n$. More importantly, given any ratio, Oresme devised a language for discussing what are now called "roots" of that ratio. Thus, since 2 : 1 is a double ratio, Oresme called that ratio, which when compounded twice with itself equals 2 : 1, half of a double ratio. (In modern terminology, this is the ratio $(2:1)^{\frac{1}{2}}$). Similarly, he called $(3:1)^{\frac{3}{4}}$ three fourth parts of a triple ratio. Oresme next developed an arithmetic for these ratios. For example, to multiply $(2:1)^{\frac{1}{3}}$ by 3 : 2, Oresme cubed the second ratio to get 27 : 8, multiplied this by 2 : 1 to get 27 : 4, and then took the cube root of the ratio considered as a fraction to get $(6\frac{3}{4})^{\frac{1}{3}}$. Similarly, to divide $(2:1)^{\frac{1}{2}}$ by 4 : 3, he divided 2 : 1 by the square of 4 : 3, namely, 16 : 9, to get 9 : 8 and then took the square root of that, $(9:8)^{\frac{1}{2}}$. In some sense, then, Oresme's works show for the first time operational rules for dealing with exponential expressions with fractional exponents.

Oresme even attempted to deal with what we would call irrational exponents. He felt intuitively that "every ratio is just like a continuous quantity with respect to division," that is, that one could take any possible "part" of such a ratio. So, "there will be some ratio which will be part of a double ratio and yet will not be half of a double nor a third part or fourth part or two-thirds part, etc., but it will be incommensurable to a double and, consequently [incommensurable] to any [ratio] commensurable to this double ratio."[25] Because Oresme had no notation for irrational exponents, he could only convey his sense of them negatively. That is, he felt that ratios of the form $(2:1)^r$ should exist even when r was not a rational number. "And further, by the same reasoning there could be some ratio incommensurable to a double and also to a triple ratio and [consequently incommensurable] to any ratios commensurable to these. . . . And there might be some irrational ratio which is incommensurable to any rational ratio. Now the reason for this seems to be that if some ratio is incommensurable to two [rational ratios] and some ratio is incommensurable to three rational ratios and so on, then there might be some ratio incommensurable to any rational ratio whatever. . . . However, I do not know how to demonstrate this."[26] What Oresme was apparently expressing, in terms of modern ideas, was that since the number line is continuous and since, for example, the fractional powers of 2 do not exhaust all (real) numbers, there must be (nonfractional) powers of 2 equal to the real numbers not already included. In fact, somewhat later in the text he states a theorem to the effect that irrational ratios are much more prevalent than rational ones:

PROPOSITION III–10 *It is probable that two proposed unknown ratios are incommensurable because if many unknown ratios are proposed it is most probable that any [one] would be incommensurable to any [other].*

Although Oresme had no formal way of proving this result, he noted that if one considers all the integral ratios from 2 : 1 up to 101 : 1, there are 4950 ways of comparing these two by two in terms of exponents (always comparing a greater ratio to a smaller), but only 25 ways with rational exponents. For example, $4:1=(2:1)^2$ and $8:1=(4:1)^{\frac{3}{2}}$. On the other hand, there is no rational exponent r such that $3:1=(2:1)^r$. Oresme then used a probability argument to conclude that astrology must be fallacious. His argument is that with great probability the ratio of any two unknown ratios, for example, those that represent various celestial motions, will be irrational. Since, therefore, there can be no exact repetitions of planetary conjunctions or oppositions, and since astrology rests on such endless repetitions, the whole basis of that "science" is false.

10.5.2 Velocity

The efforts to turn Aristotle's ideas on motion into quantitative results also resulted in new mathematics. In particular, these ideas were developed by Bradwardine and another scholar, William Heytesbury, at Merton College in the early fourteenth century. Recall that Greek mathematicians, including Autolycus and Strato, had dealt with the notion of uniform velocity and, to some extent, accelerated motion, but never considered velocity or acceleration as independent quantities that could be measured. Velocities were only dealt with by comparing distances and times, and therefore, in essence, only average velocities (over certain time periods) could be compared.

The fourteenth century, however, saw the beginning of the notion of velocity, and in particular instantaneous velocity, as measurable entities. Thus, Bradwardine in his *Tractatus de continuo* (*Treatise on the Continuum*) (c. 1330) defined the "grade" of motion as "that part of the matter of motion susceptible to 'more' and 'less.'"[27] Bradwardine then showed how to compare velocities: "In the case of two local motions which are continued in the same or equal times, the velocities and distances traversed by these [movements] are proportional, i.e., as one velocity is to the other, so the space traversed by the one is to the space traversed by the other. . . . In the case of two local motions traversing the same or equal spaces, the velocities are inversely proportional to the time, that is, as the first velocity is to the second, so the time of the second velocity is to the time of the first."[28] In other words, if two objects travel at (uniform) velocities v_1, v_2, respectively in times t_1, t_2, and cover distances s_1, s_2, then (1) if $t_1 = t_2$, then $v_1 : v_2 = s_1 : s_2$, and (2) if $s_1 = s_2$, then $v_1 : v_2 = t_2 : t_1$. Bradwardine thus considered uniform velocity itself as a type of magnitude, capable of being compared with other velocities.

Heytesbury, only a few years later in his *Regule solvendi sophismata* (*Rules for Solving Sophisms*, 1335), gave a careful definition of instantaneous velocity for a body whose motion is not uniform: "In nonuniform motion . . . the velocity at any given instant will be measured by the path which would be described by the . . . point if, in a period of time, it were moved uniformly at the same degree of velocity with which it is moved in that given instant, whatever [instant] be assigned."[29] Having given this explicit definition, Heytesbury noted by example that if two points have the same instantaneous velocity at a particular instant, they do not necessarily travel equal distances in equal times, because their velocities may well differ at other instants.

Heytesbury also dealt with acceleration in this same section: "Any motion whatever is uniformly accelerated if, in each of any equal parts of the time whatsoever, it acquires an equal increment of velocity. . . . But a motion is nonuniformly accelerated . . . when it acquires . . . a greater increment of velocity in one part of the time than in another equal part. . . . And since any degree of velocity whatsoever differs by a finite amount from zero velocity . . . , therefore any mobile body may be uniformly accelerated from rest to any assigned degree of velocity."[30] This statement provides not only a very clear definition of uniform acceleration but also, in nascent form at least, the notion of velocity changing with time. In other words, velocity is being described by Heytesbury as a "function" of time.

How does one determine the distance traveled by a body being uniformly accelerated? The answer, generally known today as the **mean speed rule**, was first stated by Heytesbury in this same work: "When any mobile body is uniformly accelerated from rest to some given degree [of velocity], it will in that time traverse one-half the distance that it would traverse if, in that same time it were moved uniformly at the [final] degree [of velocity]. . . . For that motion, as a whole, will correspond to . . . precisely one-half that degree which is its terminal velocity."[31] In modern notation, if a body is accelerated from rest in a time t with a uniform acceleration a, then its final velocity is $v_f = at$. What Heytesbury is saying is that the distance traveled by this body is $s = \frac{1}{2}v_f t$. Substituting the first formula in the second gives the standard modern formulation $s = \frac{1}{2}at^2$.

Heytesbury gave a proof of the mean speed theorem by an argument from symmetry, taking as his model a body d accelerating uniformly from rest to a velocity of 8 in one hour. (The number 8 does not represent any particular speed, but is just used as the basis for his example.) He then considered three other bodies, a moving uniformly at a speed of 4 throughout the hour, b accelerating uniformly from 4 to 8 in the first half hour, and c decelerating uniformly from 4 to 0 in that same half hour. First, he noted that body d goes as far in the first half hour as does c and as far in the second half hour as does b. Therefore, d travels as far in the whole hour as the total of b and c in the half hour. Second, he argued that since b increases precisely as much as c decreases, together they will traverse as much distance in the half hour as if they were both held at the speed of 4. This latter distance is the same that a travels in the whole hour. It follows that d goes exactly as far as does a in the hour, and the mean speed theorem is demonstrated, at least to Heytesbury's satisfaction. He then proved the easy corollary, that the body d traverses in the second half hour exactly three times the distance it covered in the first half hour.

Other scholars at Merton College in the same time period began to explore the idea of representing velocity, as well as other varying quantities, by line segments. The basic idea seems to come, in effect, from Aristotle, because such notions as time, distance, and length (of line segments) were conceived of as magnitudes in the Greek philosopher's distinction between the two types of quantities. All were infinitely divisible, and hence it was not unreasonable to attempt to represent the somewhat abstract idea of velocity, now itself being quantified, by the concrete geometric idea of a line segment. Velocities of different "degrees" would thus be represented by line segments of different lengths. Oresme carried this idea to its logical conclusion by introducing a two-dimensional representation of velocity changing with respect to time. In fact, in his *Tractatus de configurationibus qualitatum et motuum* (*Treatise on the Configuration of Qualities and Motions*) of about 1350, Oresme even generalized this idea to other cases where a given quantity varied in intensity over either distance or time. Oresme began by explaining why one can use lines to represent such quantities as velocity:

> Every measurable thing except numbers is imagined in the manner of continuous quantity. Therefore, for the mensuration of such a thing, it is necessary that points, lines, and surfaces, or their properties, be imagined. For in them, as [Aristotle] has it, measure or ratio is initially found, while in other things it is recognized by similarity as they are being referred by the intellect to them [the geometrical entities]. Although indivisible points, or lines, are nonexistent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. Therefore, every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point.[32]

From these straight lines Oresme constructed what he called a **configuration**, a geometrical figure consisting of all the perpendicular lines drawn over the base line. In the case of velocities, the base line represented time, while the perpendiculars represented the velocities at each instant. The entire figure represented the whole distribution of velocities, which

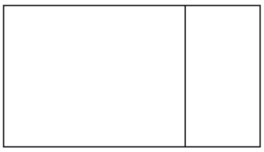


FIGURE 10.12

Uniform velocity

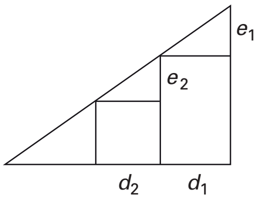


FIGURE 10.13

Uniformly difform velocity, where $d_1 : d_2 = e_1 : e_2$

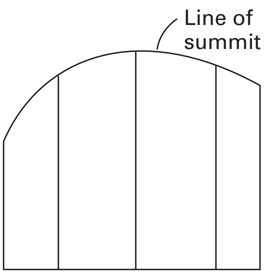


FIGURE 10.14

Difformly difform velocity, or nonuniform acceleration

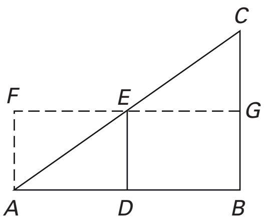


FIGURE 10.15

Proof of mean speed theorem due to Oresme

Oresme interpreted as representing the total distance traveled by the moving object. Oresme did not use what we call coordinates. There was no particular fixed length by which a given degree of velocity was represented. The important idea was only that "equal intensities are designated by equal lines, a double intensity by a double line, and always in the same way if one proceeds proportionally."[33]

For Oresme, then, a uniform quality, for example, a body moving with uniform velocity, is represented by a rectangle, for at each point the velocity is the same (Fig. 10.12). The area of the rectangle represents the total distance traveled. The distance traveled by a body beginning at rest and then moving with constant acceleration, representing what Oresme calls a "uniformly difform" quality, one whose intensity changes uniformly, is the area of a right triangle (Fig. 10.13). As Oresme noted, "A quality uniformly difform is one in which if any three points [of the subject line] are taken, the ratio of the distance between the first and the second to the distance between the second and the third is as the ratio of the excess in intensity of the first point over that of the second point to the excess of that of the second point over that of the third point, calling the first of those three points the one of greatest intensity."[34] This equality of ratios naturally defines a straight line, the hypotenuse of the right triangle. Finally, a "difformly difform" quality, such as nonuniform acceleration, is represented by a figure whose "line of summit" is a curve that is not a straight line (Fig. 10.14). In other words, Oresme in essence developed the idea of representing the functional relationship between velocity and time by a curve. In fact, he noted, "the aforesaid differences of intensities cannot be known any better, more clearly, or more easily than by such mental images and relations to figures."[35] In other words, this geometrical representation of varying quantities provided the best way to study them.

Given this representation of the motion of bodies, it was easy for Oresme to give a geometrical proof of the mean speed theorem. For if triangle ABC represents the configuration of a body moving with a uniformly accelerated motion from rest, and if D is the midpoint of the base AB, then the perpendicular DE represents the velocity at the midpoint of the journey and is half the final velocity (Fig. 10.15). The total distance traveled, represented by triangle ABC, is then equal to the area of the rectangle $ABGF$, precisely as stated by the Mertonians.

Oresme's geometric technique reappeared some 250 years later in the work of Galileo. The difference between the two lay mainly in that Galileo assumed that uniform acceleration from rest was the physical rule obeyed by bodies in free fall, while Oresme was studying the subject only abstractly. This abstraction is evident in Oresme's consideration of cases involving velocities increasing without bound. For example, he considered the case where the velocity of an object during the first half of the time interval AB, taken equal to 1 unit, is equal to 1, that in the next quarter is equal to 2, that in the next eighth is 3, in the next sixteenth 4, and so on, and proceeded to calculate the total distance traveled. In effect, he was summing the infinite series

$$\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 3+\cdots+\frac{1}{2^n}\cdot n+\cdots.$$

His result was that the sum, representing the total distance, is 2, or, as he put it, "precisely four times what is traversed in the first half of the [time]."[36] His proof, given geometrically, is very elegant. He drew a square of base CD equal to AB (=1) and divided it "to infinity

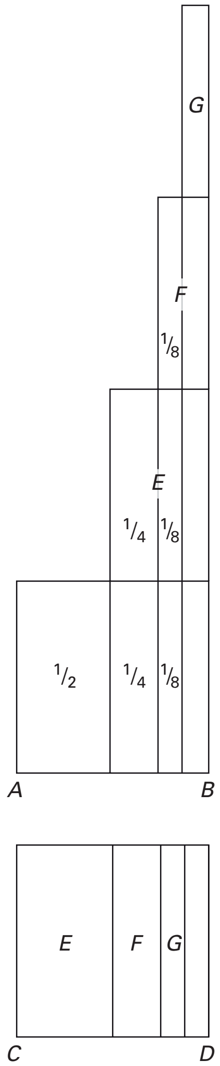


FIGURE 10.16
Oresme's summation of $\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 1 + \frac{1}{4}\cdot 3 + \cdots + \frac{1}{2^n}\cdot n + \cdots$

into parts continually proportional according to the ratio 2 to 1" (Fig. 10.16). Namely, E represents half of the square, F one-quarter, G one-eighth, and so on. The rectangle E was placed over the right half of the square on AB, F atop the new configuration over its right quarter, G atop the right eighth, and so on. It is then evident that the total area of the new configuration, which represents the total distance traveled, is not only equal to the sum of the infinite series but also equal to the sum of the areas of the two original squares.

Oresme's idea of representing velocities, as well as other qualities, geometrically, was continued in various works by others over the next century. However, no one was able to extend the representation of distances to situations more complex than Oresme's uniformly difform qualities. Eventually, even this idea was lost. Much the same fate befell the ideas of the other major European mathematicians of the medieval period. Their works were not studied and their new ideas had to be rediscovered centuries later. This lack of "progress" is evident in the stagnant mathematical curricula at the first universities as well as at the many new ones founded in succeeding centuries. With the works of Aristotle continuing to be the basis of the curriculum, the only mathematics studied was that which found its use in helping the student to understand the works of the great philosopher. Although an Oresme might carry these ideas further, such men were rare. In addition, the ravages of the Black Death and the Hundred Years War caused a marked decline in learning in France and England. It was therefore in Italy and Germany that a few of the ideas of the medieval French and English mathematicians would generate new ideas in the Renaissance.

EXERCISES

1. This problem and the next two are from Alcuin's *Propositions for Sharpening Youths*.[37] A cask is filled to 100-*metreta* capacity through three pipes. One-third of its capacity plus 6 *modii* flows in through one pipe; one-third of its capacity flows in through another pipe; but only one-sixth of its capacity flows in through the third pipe. How many *sextarii* flow in through each pipe? (Here a *metreta* is 72 *sextarii* and a *modius* is 200 *sextarii*.)
2. A man must ferry a wolf, a goat, and a head of cabbage across a river. The available boat, however, can carry only the man and one other thing. The goat cannot be left alone with the cabbage, nor the wolf with the goat. How should the man ferry his three items across the river?
3. A hare is 150 paces ahead of a hound that is pursuing him. If the hound covers 10 paces each time the hare cover 6, in how many paces will the hound overtake the hare?
4. Use Abraham bar Ḥiyya's table to find the length of the arc cut off by a chord of length 6 in a circle of diameter $10\frac{1}{2}$.
5. Find the area of the circle segment determined by the chord in Exercise 4.
6. Find the length of the chord that cuts off an arc of length $5\frac{1}{2}$ in a circle of diameter 33.
7. If a chord of length 8 has distance 2 from the circumference, find the diameter of the circle.
8. The *Artis cuiuslibet consummatio* claimed that the formula $A = \frac{3n^2-n}{2}$ gave the area of a pentagon of side n. Show, instead, that it provides a formula for the nth pentagonal number. Calculate the area of regular pentagons with sides of length $n = 1, 2, 3$, and compare your answer to the value of the $(n+1)$st pentagonal number. How close an approximation does the given formula provide?
9. Suppose you sight a tower from two stations as in Figure 10.7. At station S_1, you calculate that the ratio of height to distance d_1 is $2:5$. At station S_2, you calculate the ratio of height to distance d_2 to be $2:7$. If the distance between the two stations is 50 feet, how high is the tower?
10. Use Leonardo's table of chords to solve the following: Suppose a given chord in a circle of diameter 10 is 8 rods, 3 feet, $16\frac{2}{7}$ *unciae*. Find the length of the arc cut off by the chord.
11. From Leonardo's *Practica geometriae:* Given the quadrilateral inscribed in a circle with $ab = ag = 10$ and $bg = 12$, find the diameter ad of the circle (Fig. 10.17).

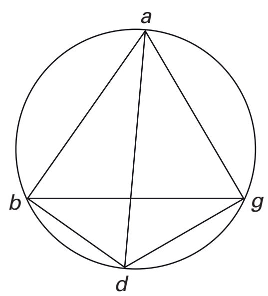


FIGURE 10.17

Determining the diameter of a circle in the work of Leonardo of Pisa

12. Develop a formula, as did Richard of Wallingford, to calculate the chord of the sum of three arcs. Translate this into a formula for the sine of the sum of three arcs.
13. Prove this theorem from Levi ben Gerson's *Trigonometry:* If all sides of any triangle whatever are known, its angles are also known. Start by dropping a perpendicular from one

vertex to the opposite side (or opposite side extended), and show how one can calculate the angles.

14. Prove the general combinatorial rule by induction:

$$C_k^n = \sum_{i=k-1}^{n-1} C_{k-1}^i.$$

15. Prove Proposition 30 from the *Maasei Ḥoshev:* $(1 + 2 + \cdots + n) + (1 + 2 + \cdots + (n - 1)) = n^2$.

16. Prove Proposition 32 of the *Maasei Ḥoshev:*

$$1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + n) = \begin{cases} 1^2 + 3^2 + \cdots + n^2 & n \text{ odd;} \\ 2^2 + 4^2 + \cdots + n^2 & n \text{ even.} \end{cases}$$

17. Prove Proposition 33 of the *Maasei Ḥoshev:*

$$(1 + 2 + 3 + \cdots + n) + (2 + 3 + \cdots + n) + (3 + \cdots + n) + \cdots + n = 1^2 + 2^2 + \cdots + n^2.$$

18. Prove Proposition 34 of the *Maasei Ḥoshev:*

$$[(1 + 2 + \cdots + n) + (2 + 3 + \cdots + n) + \cdots + n] + [1 + (1 + 2) + \cdots + (1 + 2 + \cdots + (n - 1))] = n(1 + 2 + \cdots + n).$$

19. Use the three previous results to prove the following:

$$1^2 + 2^2 + \cdots + n^2 = [n - \frac{1}{3}(n - 1)][1 + 2 + \cdots + n].$$

20. One of the problems from the *Maasei Ḥoshev:* A barrel has various holes: The first hole empties the full barrel in 3 days; the second hole empties the full barrel in 5 days; another hole empties the full barrel in 20 hours; and another hole empties the full barrel in 12 hours. All the holes are opened together. How much time will it take to empty the barrel?

21. Another problem from the *Maasei Ḥoshev:* A merchant sells four drugs. The cost of the first drug is 2 *dinars* per *litra*; the cost of the second is 3 *dinars* per *litra*; the cost of the third is 12 *dinars* per *litra*; the cost of the fourth is 20 *dinars* per *litra*. How many *litras* should one buy of each of the drugs so that the cost for each is the same?

22. Prove that the difference of the squares of two consecutive triangular numbers is a cube. (Factor the difference of squares as a sum times a difference and use the result of Exercise 15.) (This result is from Jordanus's *Arithmetica*. Jordanus uses this result, as did al-Karajī and Levi ben Gerson, to prove that the sum of the integral cubes from 1 to n is the square of the nth triangular number.)

23. Recall that Jordanus used the Pascal triangle in Proposition IX–70 of the *Arithmetica* to determine series of numbers in continued proportion. Namely, beginning with the series 1, 1, 1, 1, . . . , he derived first the series 1, 2, 4, 8, . . . , and by using those terms, he derived the series 1, 3, 9, 27, ... Now use this latter series in the same way to derive the series 1, 4, 16, 64, ... Formulate and prove by induction a generalization of this result.

24. This problem and the next six are taken from the *Liber abbaci*. One roll of saffron is sold for 3 bezants and $7\frac{1}{4}$ mils (where there are 10 mils in a bezant). How much are 17 rolls and $5\frac{1}{2}$ ounces worth (where there are 12 ounces to the roll)?

25. A Genoese *solidus* is sold for $21\frac{1}{2}$ Pisan *denarii*. How much are 7 Genoese *solidi* and 5 *denarii* worth in Pisan money? (Recall that 1 *solidus* equals 12 *denarii*.)

26. If an Imperial *solidus* is sold for $31\frac{1}{2}$ Pisan *denarii*, and a Genoese *solidus* is worth $19\frac{3}{4}$ Pisan *denarii*, then how many Genoese pounds will one have for 17 Imperial pounds, 11 *solidi*, and 5 *denarii*? (One pound equals 20 *solidi*. Note that the exchange rate between Pisan and Genoese money is different in this exercise from that stated in the previous exercise.)

27. If 7 rolls of pepper are worth 4 bezants, and 9 pounds of saffron are worth 11 bezants, how much saffron will be had for 23 rolls of pepper?

28. If a lion eats one sheep in 4 hours, a leopard eats one sheep in 5 hours, and a bear eats one sheep in 6 hours, how long would it take the three animals together to devour one sheep? (Begin by supposing that the answer is 60, the least common multiple of 4, 5, 6.)

29. Two men have some *denarii*. The first said to the second, if you will give me one of your *denarii*, then mine will equal yours. The other responded, and if you will give me one of your *denarii*, then I will have ten times as many as you. How many does each man have?

30. Solve this problem discussed in the text: There are five men with money who have found a purse with additional money. The amount the first has together with the amount in the purse is $2\frac{1}{2}$ times the total of the amounts held by the other four. Similarly, the second man's amount together with the amount in the purse is $3\frac{1}{3}$ times the total held by the others. Analogously, the fraction is $4\frac{1}{4}$ for the third man, $5\frac{1}{5}$ for the fourth man, and $6\frac{1}{6}$ for the fifth man. Find the amounts of money that each man had originally as well as the amount in the purse. (Note that Leonardo found that the first man actually had a debt of 49,154.)

31. The Fibonacci sequence (the sequence of rabbit pairs) is determined by the recursive rule $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Show that

$$F_{n+1} \cdot F_{n-1} = F_n^2 - (-1)^n$$

and that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}.$$

32. Prove that Leonardo's "congruous" numbers are always divisible by 24.

33. From the *Book of Squares:* Find a square number for which the sum of it and its root is a square number and for which the difference of it and its root is similarly a square number. (In modern notation, find x, y, z, such that $x^2 + x = z^2$ and $x^2 - x = y^2$. Leonardo began his solution by using the congruous number 24 in solving $a^2 + 24 = b^2$, $a^2 - 24 = c^2$; he then divided everything by 24.)

34. This problem and the next two are from Jordanus's *De numeris datis*. If the sum of the product of the two parts of a given number and of their difference is known, then each of them is determined. Namely, solve the system $x + y = a$, $xy + x - y = b$. Use Jordanus's example where $a = 9$ and $b = 21$.

35. If the sum of the two quotients formed by dividing the two parts of a given number by two different known numbers is given, then each of the parts is determined. Namely, solve the system $x + y = a$, $x/b + y/c = d$. Jordanus sets $a = 10$, $b = 3$, $c = 2$, and $d = 4$.

36. If the sum of two numbers is given together with the product of their squares, then each of them is determined. Jordanus's example is $x + y = 9$, $x^2y^2 = 324$.

37. Use Oresme's technique to divide a sesquialterate ratio $(3:2)$ by a third part of a double ratio $(2:1)^{\frac{1}{3}}$.

38. Show that there are in fact 4950 ways of comparing (by ratio) the 100 integral ratios from 2 : 1 up to 101 : 1 and that precisely 25 will have rational exponents.

39. Show that under the assumptions of the mean speed theorem, if one divides the time interval into four equal subintervals, the distances covered in each interval will be in the ratio 1 : 3 : 5 : 7. Generalize this statement to a division of the time interval into n equal subintervals and prove your result.

40. From Oresme's *Tractatus de configurationibus qualitatum et motuum*: Show geometrically that the sum of the series

$$48 \cdot 1 + 48 \cdot \frac{1}{4} \cdot 2 + 48 \cdot \left(\frac{1}{4}\right)^2 \cdot 4 + \cdots + 48\left(\frac{1}{4}\right)^n \cdot 2^n + \cdots$$

is equal to 96.

41. Solve the following problem of Oresme: Divide the line AB of length 1 (representing time) proportionally to infinity in a ratio of 2 : 1; that is, divide it so the first part is one-half, the second one-quarter, the third one-eighth, and so on. Let there be a given finite velocity (say, 1) in the first interval, a uniformly accelerated velocity (from 1 to 2) in the second, a constant velocity (2) in the third, a uniformly accelerated velocity (from 2 to 4) in the fourth, and so on (Fig. 10.18). Show that the total distance traveled is 7/4.

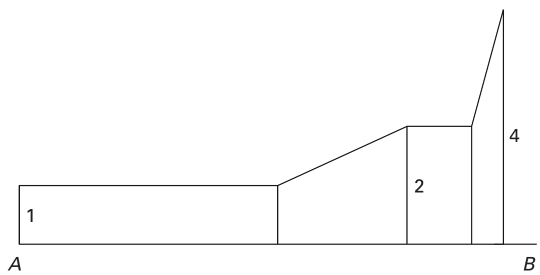


FIGURE 10.18
A problem of Oresme

42. Prove the result of Oresme: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ becomes infinite. (This series is usually called the harmonic series.)

43. Determine what mathematics was necessary to solve the Easter problem. What was the result of the debate in the Church? How is the date of Easter determined today? (Note that the procedure in the Roman Catholic Church is different from that in the Eastern Orthodox Church.)

44. Compare Levi ben Gerson's use of "induction" to that of al-Karajī. Should the methods of either be considered "proof by induction"? Discuss.

45. Write a lesson demonstrating proof by induction using some of Levi ben Gerson's examples.

46. Write a lesson developing some of the basic combinatorial rules using the methods of Abraham ibn Ezra and Levi ben Gerson.

47. Explain in detail why the area of one of Oresme's configurations should represent the total distance traveled by a moving object.

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