STANDARD ERROR + CENTRAL LIMIT THEOREM

(... and why we *really* should care...)

Håkon K. Gjessing

Professor/Principal Investigator

Centre for Fertility and Health, Norwegian Institute of Public Health, Oslo

Department of Global Public Health and Primary Care, University of Bergen

Makerere

Wednesday, 7 June 2023

SALMONELLA INFECTIONS

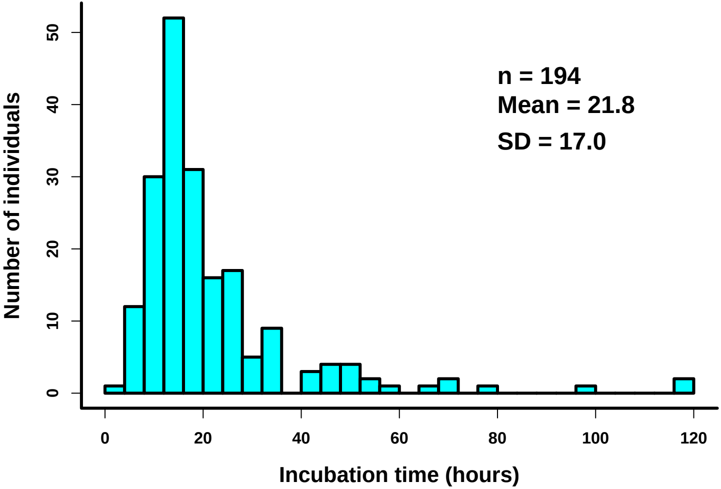
Incubation times for 194 cases of Salmonella typhimurium at a medical conference in Wales, 1986

(In hours, sorted by length):

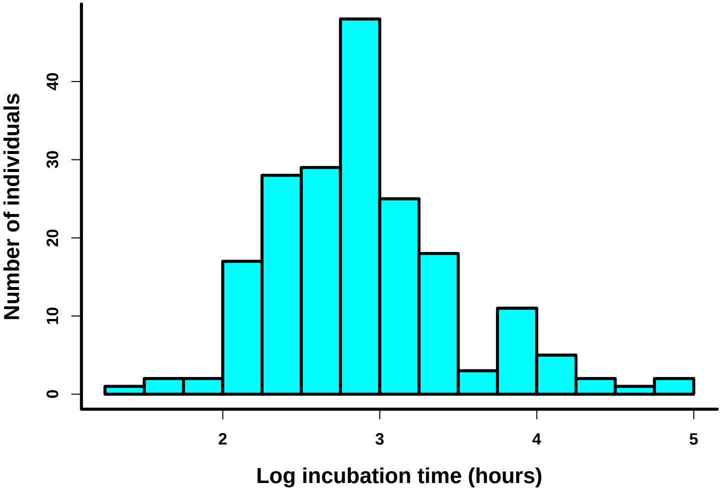
```
  [1]   3.5   4.6   4.7   6.8   7.1   7.4   7.4   7.6   7.7
 [10]   7.7   7.7   7.8   7.8   8.1   8.2   8.2   8.4   8.6
 [19]   8.6   8.9   9.0   9.2  10.6  10.9  11.1  11.2  11.3
 [28]  11.4  11.5  11.5  11.5  11.6  11.6  11.6  11.6  11.6
 [37]  11.7  11.8  11.8  11.9  11.9  11.9  11.9  12.0  12.0
 [46]  12.1  12.1  12.1  12.1  12.1  12.3  12.3  12.4  12.4
 [55]  12.5  12.5  12.5  12.5  12.6  12.9  13.0  13.2  14.7
 [64]  14.8  14.9  15.0  15.0  15.1  15.2  15.4  15.4  15.4
 [73]  15.4  15.5  15.5  15.6  15.6  15.6  15.6  15.7  15.7
 [82]  15.7  15.7  15.7  15.8  15.8  15.8  15.8  15.8  15.8
 [91]  15.8  15.9  15.9  15.9  15.9  16.0  16.0  16.0  16.1
[100]  16.1  16.2  16.2  16.3  16.3  16.3  16.3  16.4  16.5
[109]  16.6  16.6  16.6  16.7  16.7  16.7  16.8  17.0  17.3
[118]  18.4  19.4  19.4  19.4  19.5  19.5  19.7  19.9  19.9
[127]  20.0  20.1  20.3  20.3  20.4  20.5  20.7  21.0  23.2
[136]  23.3  23.3  23.5  23.8  23.8  23.9  23.9  24.0  24.2
[145]  24.2  24.2  24.3  24.3  24.4  24.4  24.6  25.3  27.0
[154]  27.1  27.1  27.5  27.5  27.5  27.8  28.1  28.3  30.8
[163]  31.5  31.7  32.0  32.1  32.1  32.1  32.1  32.3  34.9
[172]  35.4  35.7  43.4  43.6  43.8  47.3  47.7  47.7  47.9
[181]  48.0  48.1  48.1  48.4  55.0  55.8  56.1  67.9  68.7
[190]  71.1  79.9  96.2 116.0 116.2
```

CENTRE FOR FERTILITY AND HEALTH

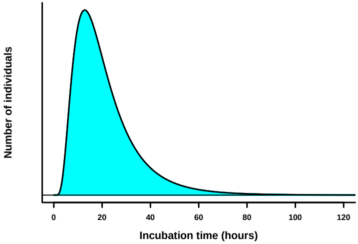
Histogram of incubation times for 194 individuals



HISTOGRAM OF *log* INCUBATION TIMES FOR 194 INDIVIDUALS



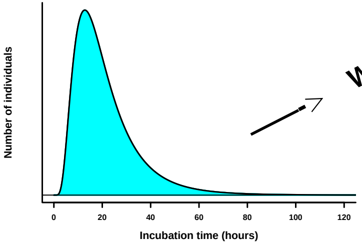
Repeated sampling from a population



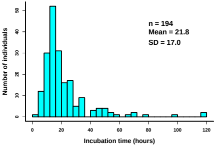
Population (unobserved)

Mean $= \mu$

Std. dev. $= \sigma$



Wales, 1986 ($n = 194$)

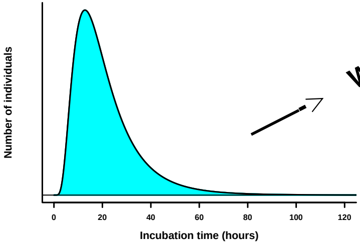


Mean $= \overline{X}_1 = 21.8$

Population (unobserved)

Mean $= \mu$

Std. dev. $= \sigma$

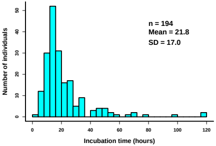


Population (unobserved)

Mean $= \mu$

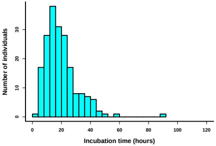
Std. dev. $= \sigma$

Wales, 1986 ($n = 194$)



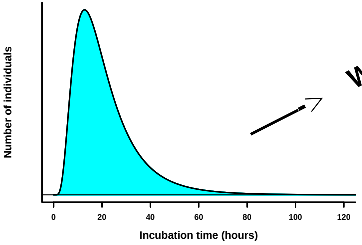
Mean $= \overline{X}_1 = 21.8$

($n = 194$)



Mean $= \overline{X}_2 = 19.7$

CENTRE FOR FERTILITY AND HEALTH

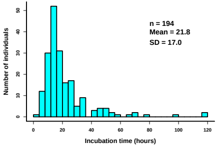


Population (unobserved)

Mean = μ

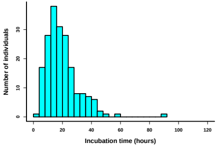
Std. dev. = σ

Wales, 1986 ($n = 194$)



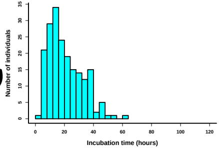
Mean = $\overline{X}_1 = 21.8$

($n = 194$)



Mean = $\overline{X}_2 = 19.7$

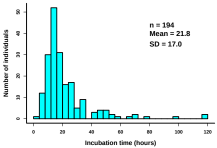
($n = 194$)



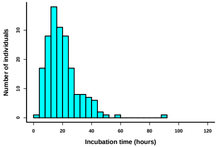
Mean = $\overline{X}_3 = 20.5$

CENTRE FOR FERTILITY AND HEALTH

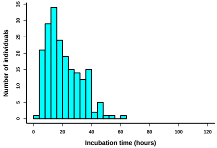
HISTOGRAM OF *averages*



Mean $= \overline{X}_1 = 21.8$



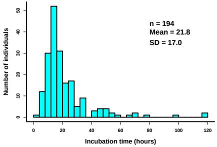
Mean $= \overline{X}_2 = 19.7$



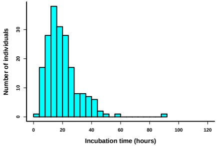
Mean $= \overline{X}_3 = 20.5$

CENTRE FOR FERTILITY AND HEALTH

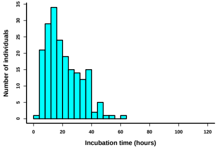
HISTOGRAM OF *averages*



Mean $= \overline{X}_1 = 21.8$



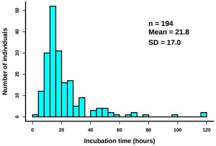
Mean $= \overline{X}_2 = 19.7$



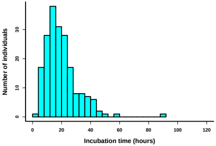
Mean $= \overline{X}_3 = 20.5$

CENTRE FOR FERTILITY AND HEALTH

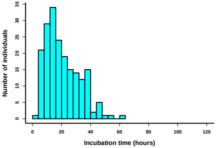
HISTOGRAM OF *averages*



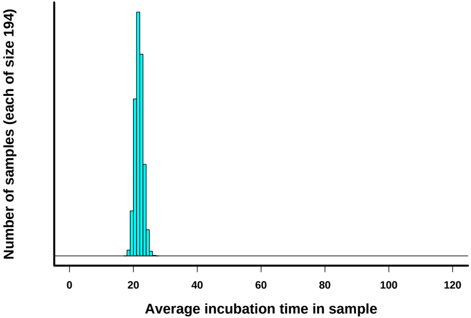
Mean $= \overline{X}_1 = 21.8$



Mean $= \overline{X}_2 = 19.7$

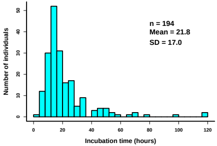


Mean $= \overline{X}_3 = 20.5$

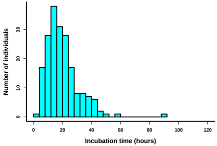


Mean $\approx \mu$

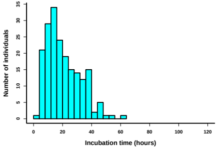
HISTOGRAM OF *averages* (ZOOMED)



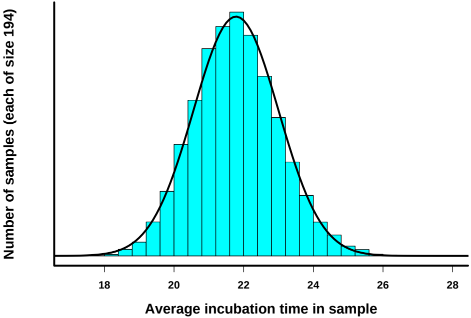
Mean $= \overline{X}_1 = 21.8$



Mean $= \overline{X}_2 = 19.7$



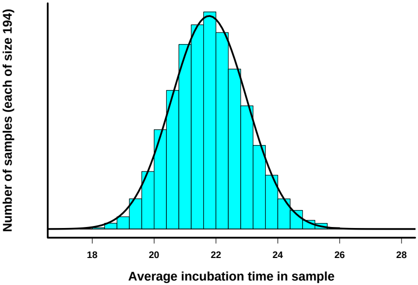
Mean $= \overline{X}_3 = 20.5$



Mean $\approx \mu$

LAW OF LARGE NUMBERS

Histogram of $\overline{X}$:



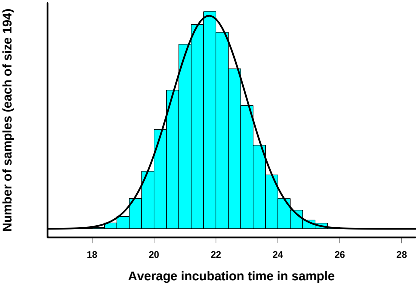
$\overline{X}$ (sample) "close to" μ (population)

- Standard deviation in ***this*** distribution: Standard Error of $\overline{X}$

- Precision $=$ SE$(\overline{X})$

LAW OF LARGE NUMBERS

Histogram of $\overline{X}$:



$\overline{X}$ (sample) "close to" μ (population)

- Standard deviation in ***this*** distribution: Standard Error of $\overline{X}$

- Precision $= \text{SE}(\overline{X})$

IMPORTANT!

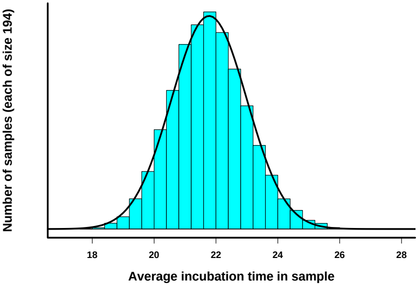
$$\text{SE}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \approx \frac{\text{SD}(X)}{\sqrt{n}}$$

$$\text{SE}(\overline{X}) \approx \frac{17.0}{\sqrt{194}} = 1.22$$

CENTRE FOR FERTILITY AND HEALTH

LAW OF LARGE NUMBERS

Histogram of $\overline{X}$:



$\overline{X}$ (sample) "close to" μ (population)

- Standard deviation in ***this*** distribution:
  Standard Error of $\overline{X}$

- Precision $= \text{SE}(\overline{X})$

IMPORTANT!

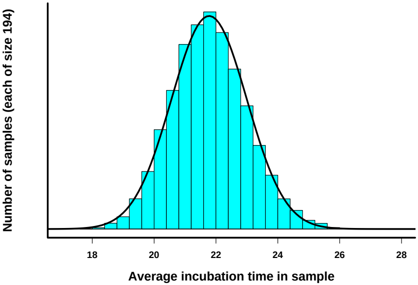
$$\text{SE}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \approx \frac{\text{SD}(X)}{\sqrt{n}}$$

$$\text{SE}(\overline{X}) \approx \frac{17.0}{\sqrt{194}} = 1.22$$

- $\sqrt{194} \approx 14$:
  $\overline{X}$ is about 14 times more precise than X

Law of Large Numbers

Histogram of $\overline{X}$:



$\overline{X}$ (sample) "close to" μ (population)

- Standard deviation in ***this*** distribution:
  Standard Error of $\overline{X}$

- Precision $= \text{SE}(\overline{X})$

Important!

$$\text{SE}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \approx \frac{\text{SD}(X)}{\sqrt{n}}$$

$$\text{SE}(\overline{X}) \approx \frac{17.0}{\sqrt{194}} = 1.22$$

- $\sqrt{194} \approx 14$:
  $\overline{X}$ is about 14 times more precise than X

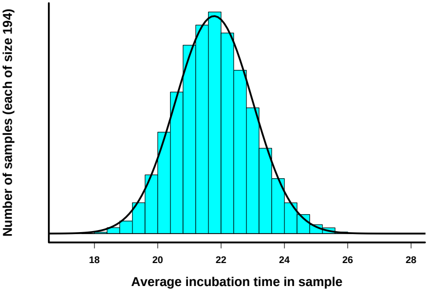
Law of large numbers:

$$\overline{X} \to \mu \quad \text{when} \quad n \to \infty$$

CENTRE FOR FERTILITY AND HEALTH

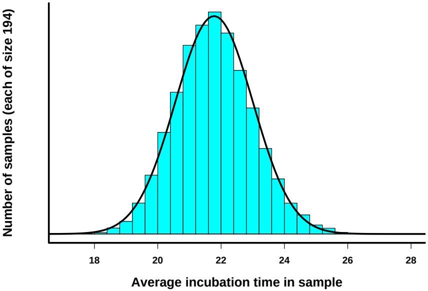
Central Limit Theorem

Histogram of $\overline{X}$:



$\overline{X}$ (sample) "close to" μ (population)

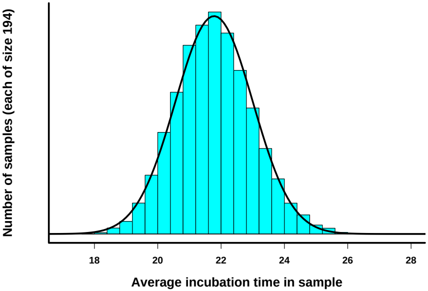
Histogram of $\overline{X}$:



$\overline{X}$ (sample) "close to" μ (population)

- Shape of distribution of $\overline{X}$:
  Gaussian (Normal) distribution
  (almost, at least)

Histogram of $\overline{X}$:



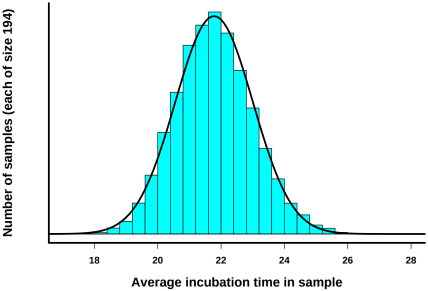
$\overline{X}$ (sample) "close to" μ (population)

- Shape of distribution of $\overline{X}$:
  Gaussian (Normal) distribution
  (almost, at least)

- Central Limit Theorem:
  Regardless of distribution of X,
  distribution of $\overline{X}$ close to Normal

CENTRE FOR FERTILITY AND HEALTH

Histogram of $\overline{X}$:

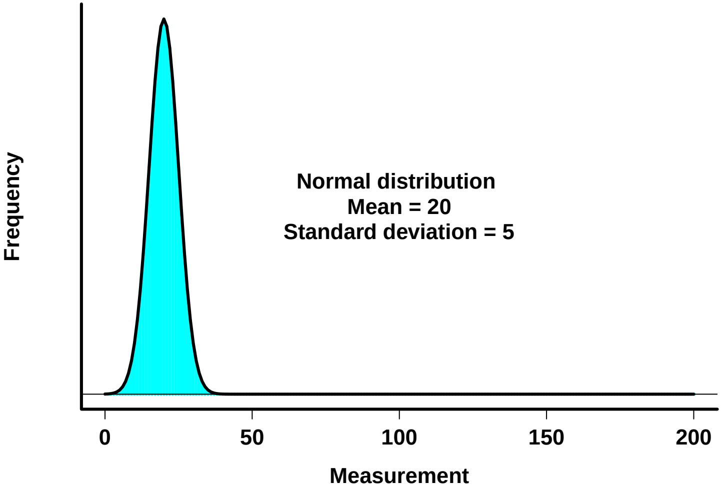


$\overline{X}$ (sample) "close to" μ (population)

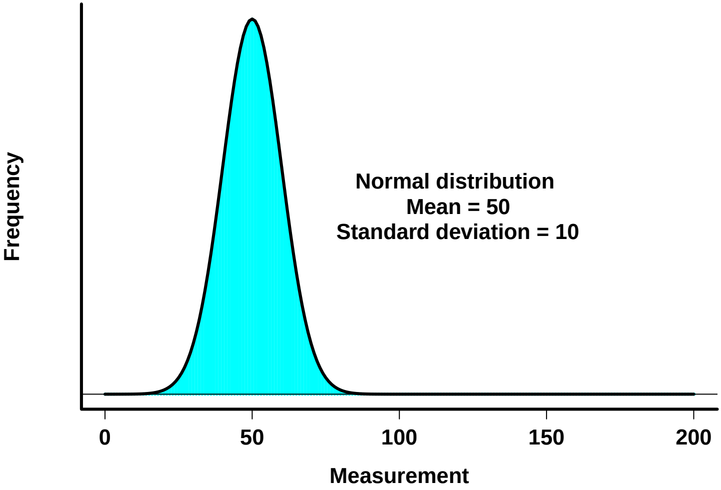
- Shape of distribution of $\overline{X}$:
  Gaussian (Normal) distribution
  (almost, at least)
- Central Limit Theorem:
  Regardless of distribution of X,
  distribution of $\overline{X}$ close to Normal
- Larger n (here 194) means
  closer to Normal
  The more data the better!

CENTRE FOR FERTILITY AND HEALTH

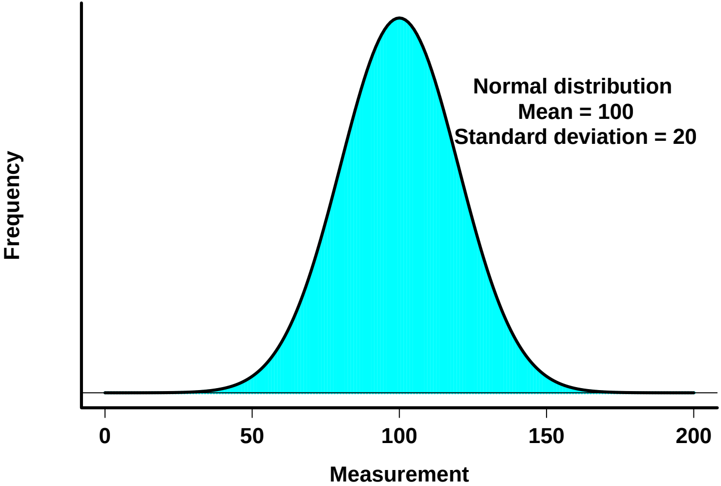
THE NORMAL (GAUSSIAN) DISTRIBUTION



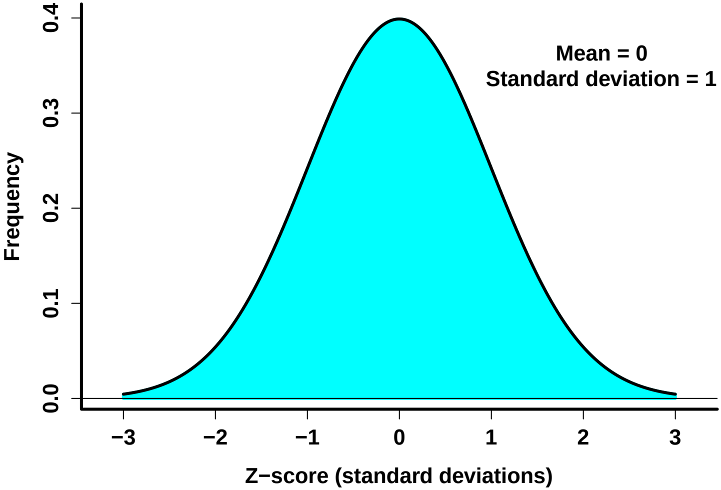
THE NORMAL (GAUSSIAN) DISTRIBUTION



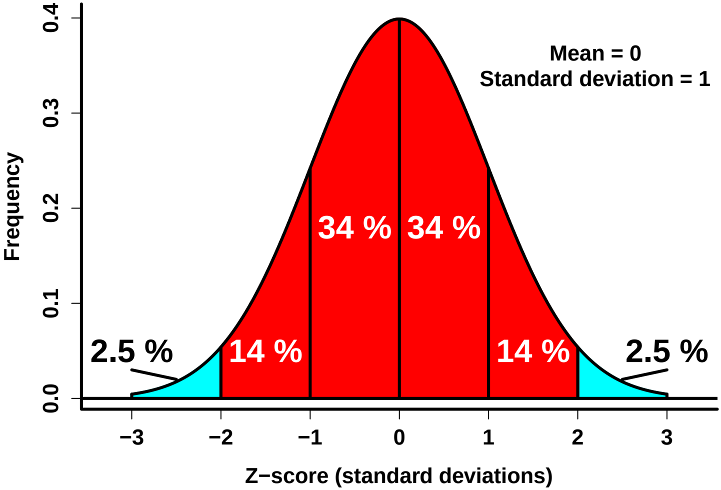
The normal (Gaussian) distribution



THE *standard* NORMAL DISTRIBUTION



The *standard* normal distribution



STATISTICAL INFERENCE

Sample		Population
$\overline{X}$	$\rightarrow$	μ
SD(X)	$\rightarrow$	σ

- The standard error

$$\text{SE}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \approx \frac{\text{SD}(X)}{\sqrt{n}}$$

is the uncertainty (precision) of $\overline{X}$ when estimating μ

How should we use this uncertainty measure??

CENTRE FOR FERTILITY AND HEALTH

Confidence interval based on the Normal distribution:

- $\overline{X}$ is approximately normally distributed ... so the interval

$$(\overline{X} - 1.96 \cdot \text{SE}(\overline{X}), \quad \overline{X} + 1.96 \cdot \text{SE}(\overline{X}))$$

has a 95% chance of covering the "true" μ.

Lower limit:

$$\overline{X} - 1.96 \cdot \text{SE}(\overline{X}) = 21.8 - 1.96 \cdot 1.22 = 19.4$$

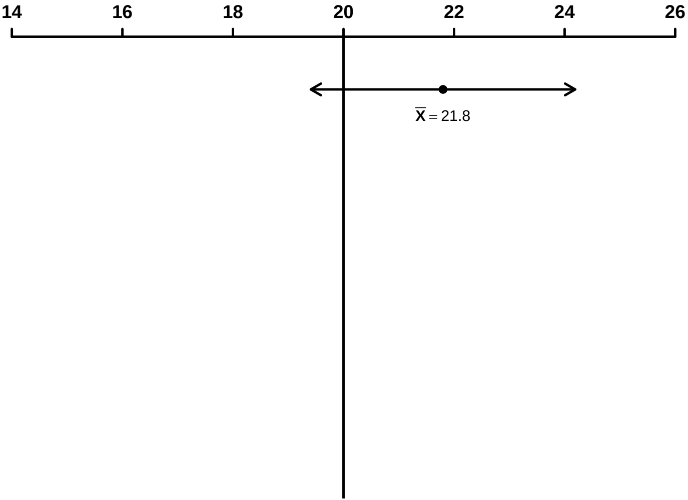
Upper limit:

$$\overline{X} + 1.96 \cdot \text{SE}(\overline{X}) = 21.8 + 1.96 \cdot 1.22 = 24.2$$

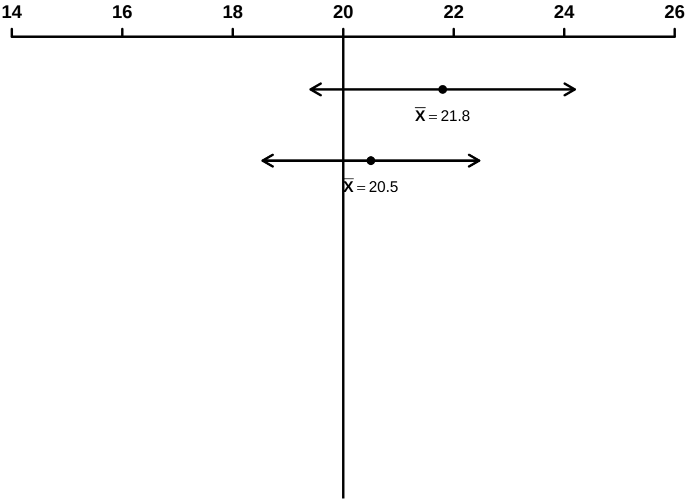
95% confidence interval: (19.4, 24.2)

CENTRE FOR FERTILITY AND HEALTH

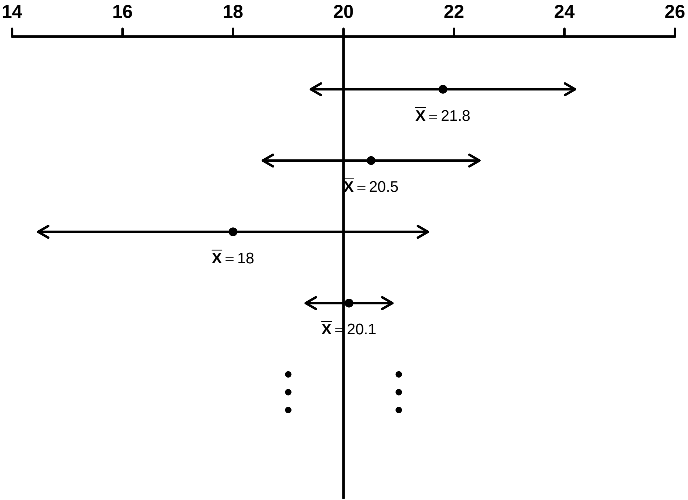
Confidence intervals in repeated experiments (95% CI)



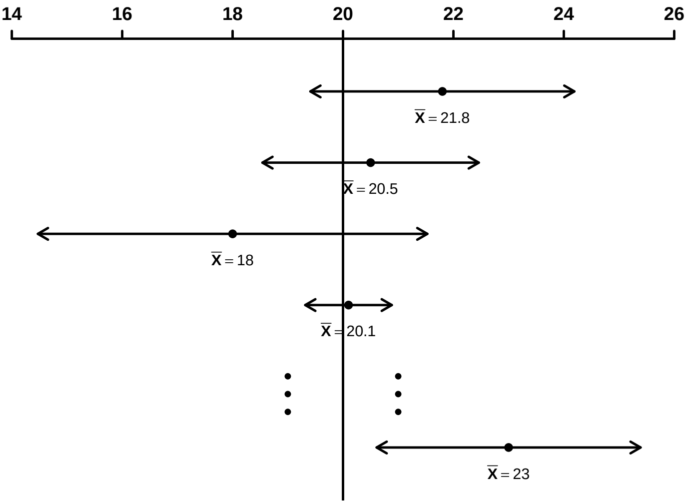
CONFIDENCE INTERVALS IN REPEATED EXPERIMENTS (95% CI)



Confidence intervals in repeated experiments (95% CI)

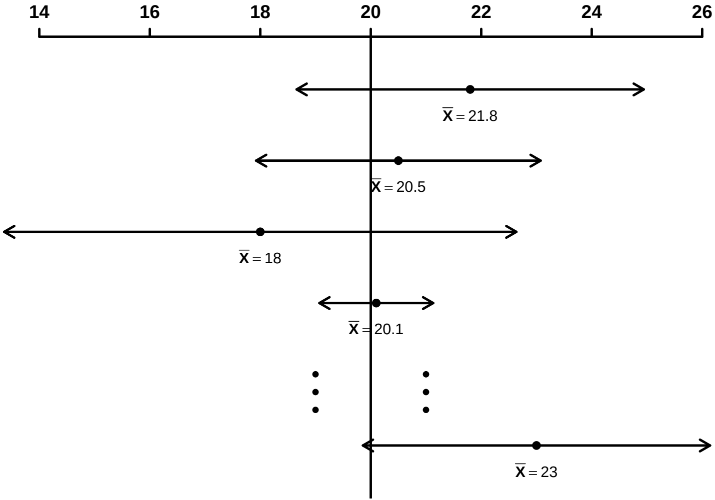


Confidence intervals in repeated experiments (95% CI)



CENTRE FOR FERTILITY AND HEALTH

Confidence intervals in repeated experiments (99% CI)



90% CI: 1.64, 95% CI: 1.96, 99% CI: 2.58

CENTRE FOR FERTILITY AND HEALTH

Special Note on the Binomial Distribution

X has a binomial distribution $\text{Bin}(n, p)$

$$X \sim \text{Bin}(n, p)$$

Expected value of X:

$$\mu = E(X) = n \cdot p$$

Standard deviation of X:

$$\sigma = SD(X) = \sqrt{n \cdot p \cdot (1 - p)}$$

Estimate of p

$$\widehat{p} = \frac{X}{n}$$

Standard error of $\widehat{p}$:

$$SE(\widehat{p}) \approx \sqrt{\frac{\widehat{p} \cdot (1 - \widehat{p})}{n}}$$

CENTRE FOR FERTILITY AND HEALTH

NOTE

A note on notation:

Textbook:

- Population proportion: π
- Sample estimate of proportion: p

Very common in other books (more or less the standard):

- Population proportion: p
- Sample estimate of proportion: $\widehat{p}$

CENTRE FOR FERTILITY AND HEALTH