Study Report 1 (6/4)

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1 Introduction

This study report presents a Renormalization Group Equation (RGE) analysis of gauge coupling constants across various energy scales. We begin by exploring the conceptual origins of the RGE, explaining why the coupling constant α, which intuitively seems like a fixed parameter, must depend on the experimental energy scale. Then, we derive Yang-Mills theory from continuous local symmetries, utilizing fundamental gauge theory frameworks. To solve the issue of gauge degeneracy within the path integral formalism, we include Fadeev-Popov ghost fields to systematically cancel redundant gauge degrees of freedom. Based on the final Lagrangian, excluding the scalar Higgs sector, we evaluate the beta function with Feynman analysis. Finally, using the derived expressions, we compare the RGE running of a toy QED model against QCD and discuss the physical phenomena driven by asymptotic freedom.

The theoretical framework and analytical derivations presented in this paper are based on the following foundational texts: Chapters 12, 15, and 16 of Peskin and Schroeder's *An Introduction to Quantum Field Theory*; Chapters 14, 28, 29, 62, 71, 72, and 73 of Srednicki's *Quantum Field Theory*; and Chapter 1 of Manohar and Wise's *Heavy Quark Physics*.

2 Renormalization Group Equation

One of the fundamental objectives of quantum field theory is to compute physical observables, such as scattering amplitudes, which must yield finite values and be independent of any arbitrary energy scale. Consider the toy model Quantum Electrodynamics (QED) as the prototype for gauge theories. The bare Lagrangian, denoted by the subscript 0, is written as:

$$\mathcal{L} = e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu} + \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - \frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} \tag{1}$$

Though the bare fields $(A_{0\mu}, \psi_0)$ and parameters (m_0, e_0) are independent of any arbitrary energy scale, they are infinite and thus unmeasurable. The physical, finite quantities are extracted from the S-matrix, which is the probability between the initial and final states:

$$\mathcal{M} = \langle f|i\rangle \tag{2}$$

To systematically absorb these infinities and calculate the physical quantities, we rescale the bare quantities and introduce renormalized fields and parameters, with their corresponding renormalization constants (Z-factors): $A_{0\mu} = Z_3^{1/2} A_\mu$, $\psi_0 = Z_2^{1/2}\psi$, $m_0 = Z_m m$, and the coupling relationship $e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} e$. The Lagrangian is then rewritten in terms of renormalized quantities:

$$\mathcal{L} = Z_1 e\bar{\psi}\gamma^\mu \psi A_\mu + Z_2\bar{\psi}(i\partial\!\!\!/ - Z_m m)\psi - \frac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} \tag{3}$$

The procedure of mathematically isolating and parameterizing the divergent integrals is called *regularization*. While energy cut-off (Λ) regularization is physically intuitive, dimensional regularization (DIMREG) is strictly required for gauge theories like QED because it preserves both Lorentz and gauge invariance. In DIMREG, the integration measure is analytically continued to $d = 4-\epsilon$ dimensions. Consequently, to keep the renormalized gauge coupling e strictly dimensionless in d dimensions, an arbitrary mass scale μ must be introduced, as explicitly shown by the $\mu^{\epsilon/2}$ factor. Once regularized, the infinities manifest as powers of $1/\epsilon$ poles. Renormalization is the specific choice of how to absorb these poles into the Z factors. The general form of a perturbative Z factor is:

$$Z = 1 + \frac{A}{\epsilon} + B. \tag{4}$$

Here A, B are finite numbers. Since finite shift doesn't effect the absorption of the infinities, the ambiguity lies in B. The choice of this finite shift defines the *renormalization scheme*. In the Modified Minimal Subtraction ($\overline{MS}$) scheme, B is chosen to absorb exactly the $1/\epsilon$ pole along with the universally accompanying $\gamma_E - \ln(4\pi)$ constants.

Since the bare parameters are completely independent of the arbitrary regularization scale μ, the renormalized quantities (such as e and m) must depend on μ to compensate for the dependence. These parameters become running variables, and their evolution is governed by the Renormalization Group Equation (RGE).

One crucial expression of RGE is a beta function $\beta(e)$, which dictates the specific flow of the coupling constant $e(\mu)$ with respect to μ:

$$\beta(e) = \frac{\partial e}{\partial \log \mu} \tag{5}$$

For Quantum Electrodynamics (QED), the 1-loop vacuum polarization correction yields a positive beta function, $\beta(e) = e^3/(12\pi^2)$. Solving this differential equation for $\alpha(\mu) \equiv e^2(\mu)/4\pi$ gives the running coupling:

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 - \frac{2\alpha(\mu_0)}{3\pi}\ln\left(\frac{\mu}{\mu_0}\right)} \tag{6}$$

The arbitrary scale μ must never chosen arbitrarily in practice. For example, in a two-particle scattering process with center-of-mass energy squared s, higher-order loop corrections inevitably generate large logarithmic terms of the form $\alpha^n \ln^n(s/\mu^2)$. To prevent the breakdown of perturbation theory, one must set μ^2 close to s. This strategic choice reduces the large logarithms into the running coupling itself, effectively making μ the energy scale of the interaction.

Concurrently, from the uncertainty principle, probing a system at a high momentum scale μ is mathematically equivalent to resolving its structure at a short distance $r \sim 1/\mu$. Thus, we can interpret RGE as an energy scale or distance scale dependence of coupling constants.

The evolution of the Renormalization Group (RG) can be understood through the viewpoint of Kenneth Wilson's renormalization theory. In Wilson's framework, an effective theory at a low-energy scale is explicitly derived by integrating out high-momentum degrees of freedom down to a cutoff scale Λ. The Λ dependency of the effective theory accounts for the evolution of the theory with the parameter μ. (To make the momentum space compact, a Wick rotation to Euclidean space is performed in advance.)

$$Z(J) = \int \mathcal{D}\varphi e^{-S_E(\varphi) - \int J\varphi} = \int \mathcal{D}\varphi_{|k|<\Lambda} e^{-S_{\text{eff}}(\varphi;\Lambda) - \int J\varphi} \tag{7}$$

$$e^{-S_{\text{eff}}(\varphi;\Lambda)} = \int \mathcal{D}\varphi_{|k|>\Lambda} e^{-S_E(\varphi)} \tag{8}$$

Lastly, there is one more important insight in the Wilson's renormalization theory. The resulting effective Lagrangian inevitably contains an infinite series of higher-dimensional local operators, $\sum c_d(\Lambda)\mathcal{O}_d$. The crucial insight of Wilsonian RG is the dimensional scaling of these operators: operators with mass dimension $d > 4$ (which is classified as non-renormalizable) are dynamically suppressed by powers of $(E/\Lambda)^{d-4}$. At low energies ($E \ll \Lambda$), these operators become irrelevant and vanish. Therefore, at leading order, the low-energy dynamics are heavily dominated by the finite number of renormalizable (marginal and relevant) terms. This assures us of including only renormalizable terms in the Lagrangian to expect new theories.

3 Yang-Mills Theory

Continuous internal symmetries dictate the fundamental interaction structure of the theory. Continuous internal symmetry implies that the Lagrangian is invariant under a specific transformation group (or gauge group) G acting on the fields. For the Standard Model, G is a continuous Lie group. An infinitesimal group element $g(\theta) \in G$ can be expanded to the first order in the transformation parameters θ^a using the generators T^a of its associated Lie algebra $\mathfrak{g}$:

$$g(\theta) = 1 + i\theta^a T^a + \mathcal{O}(\theta^2) \tag{9}$$

The Lie algebra $\mathfrak{g}$ is a vector space endowed with a non-commutative structure, defined by its Lie bracket (commutator) and the antisymmetric structure constants f^{abc}:

$$[T^a, T^b] = i f^{abc} T^c \tag{10}$$

Since the physical fields form a vector space (specifically Fork space), the abstract group elements of G must be mapped to linear operators acting on this space, a concept known as a group representation.

Let $\varphi_i(x)$ be an n-component multiplet of fields transforming under a specific representation of G, where i, j are the internal representation indices and T^a_{ij} are the corresponding $n \times n$ matrix generators. To construct a gauge theory, we promote the global symmetry to a local symmetry by allowing the transformation parameters to depend on spacetime coordinates $\theta^a \to \theta^a(x)$. The fields transform exactly as:

$$\psi_i^\alpha(x) \longrightarrow \left[\exp\left(i\theta^a(x)T^a\right)\right]_{ij} \psi_j^\alpha(x) \tag{11}$$

Note that this internal gauge transformation must be strictly distinguished from spacetime (Lorentz) transformations. The Lorentz transformation acts on the spacetime coordinates $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ globally, whereas the internal gauge transformation leaves the spacetime coordinates invariant and rotates the internal degrees of freedom locally.

$$\varphi^\alpha(x) \to M^\alpha_\beta(\Lambda)\varphi^\beta(\Lambda^{-1}x) \tag{12}$$

Finally, a single Lie group can admit multiple distinct representations. The internal representation of the non-abelian gauge group is characterized by the dimensionality of the internal vector space. Completely independently, the Lorentz representation of a field is determined by its spin and chirality.

Pure Yang-Mills theory describes the self-interacting dynamics of non-Abelian gauge bosons. When coupled to matter, it defines the interactions of fermionic fields charged under the gauge group G. To fully appreciate the mathematical rigidity of this structure, we write down the Lagrangian of a Yang-Mills field coupled to a Dirac fermion, explicitly restoring every single index. We define the following index conventions:

- $\mu, \nu \in \{0, 1, 2, 3\}$: Spacetime indices.
- $\alpha, \beta \in \{1, 2, 3, 4\}$: Dirac spinor indices. (Index of each spinor)
- $i, j \in \{1, \ldots, N\}$: Internal gauge indices for the fundamental representation. (Tuple elements of Dirac spinor)
- $a, b, c \in \{1, \ldots, N^2 - 1\}$: Internal gauge indices for the gauge group. (Lie algebra generators)

The fully indexed invariant Lagrangian is formulated as (As mentioned, it is justified to include only renormalizable terms from Wilson's reasoning):

$$\mathcal{L} = \mathcal{L}_{Matter} + \mathcal{L}_{YM} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 \tag{13}$$

where,

$$\bar{\psi}_{i\alpha} = (\psi^\dagger)_{i\beta}(\gamma^0)^\beta{}_\alpha \tag{14}$$

$$\bar{\psi}(i\not{D} - m)\psi = \bar{\psi}_{i\alpha}\left[i(\gamma^\mu)^\alpha{}_\beta(D_\mu)_{ij} - m\delta^\alpha{}_\beta\delta_{ij}\right]\psi^\beta_j \tag{15}$$

$$(F^a_{\mu\nu})^2 = \eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ab}F^a_{\mu\nu}F^b_{\rho\sigma} \tag{16}$$

Here, $\eta^{\mu\rho}$ is the Minkowski metric, and δ_{ab}, δ_{ij}, δ^α_β are Kronecker deltas ensuring the invariant contraction of the respective spaces.

The covariant derivative $(D_\mu)_{ij}$ acting on the fundamental representation is explicitly:

$$(D_\mu)_{ij} = \partial_\mu\delta_{ij} - igA^a_\mu(T^a)_{ij} \tag{17}$$

where A^a_μ is the gauge boson field, g is the universal gauge coupling constant, and $(T^a)_{ij}$ are the $N \times N$ matrix elements of the group generators. The gauge boson field plays a crucial role for the Lagrangian being invariant even though the transformation is local. The mandatory infinitesimal transformation rule for the gauge boson field is below.

$$A^a_\mu(x) \to A^a_\mu(x) - [\delta^{ac}\partial_\mu - igA^b_\mu(T^b)^{ac}]\theta^c(x) \equiv A^a_\mu(x) - D^{ac}_\mu\theta^c(x) \tag{18}$$

The non-Abelian field strength tensor $F^a_{\mu\nu}$ is derived from the commutator of the covariant derivatives $[D_\mu, D_\nu]$, expanding to:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{19}$$

The current Standard Model is known to have $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry. The following is the table of each fermionic field, gauge boson fields, index convention, and every transformation rule for each index. Since it is a local symmetry, θ is dependent on the spacetime coordinate.

Field	**Physical Meaning**	**Fully Indexed Form**
Q_L	Left-handed Quark Doublet	$(Q_L)^m_{iI\alpha}$
u_R	Right-handed Up-type Quark	$(u_R)^m_{i\dot{\alpha}}$
d_R	Right-handed Down-type Quark	$(d_R)^m_{i\dot{\alpha}}$
L_L	Left-handed Lepton Doublet	$(L_L)^m_{I\alpha}$
e_R	Right-handed Lepton	$(e_R)^m_{\dot{\alpha}}$

Table 1: Fermion Field

Field	Physical Meaning	Fully Indexed Form
G	Gluon ($SU(3)_c$)	G^a_μ
W	Weak Isospin Gauge Field ($SU(2)_L$)	W^A_μ
B	Hypercharge Gauge Field ($U(1)_Y$)	B_μ

Table 2: Gauge Boson Field

Index	Range (Set)	Meaning / Associated Space
μ, ν	$\{0, 1, 2, 3\}$	Spacetime (Lorentz Vector Index)
α, β	$\{1, 2\}$	Left-handed Weyl Spinor Space (Undotted)
$\dot{\alpha}, \dot{\beta}$	$\{1, 2\}$	Right-handed Weyl Spinor Space (Dotted)
m, n	$\{1, 2, 3\}$	Generation / Flavor Space)
i, j	$\{1, 2, 3\}$	$SU(3)_c$ Fundamental Representation (Color)
a, b, c	$\{1, \ldots, 8\}$	$SU(3)_c$ Adjoint Representation (Gluon Color)
I, J	$\{1, 2\}$	$SU(2)_L$ Fundamental Representation (Weak Isospin)
A, B, C	$\{1, 2, 3\}$	$SU(2)_L$ Adjoint Representation (W Boson Isospin)

Table 3: Index Convention

Index Type	Infinitesimal Transformation ($\psi \to \psi + \delta\psi$)
i ($SU(3)_c$)	$\psi_i(x) \to \psi_i(x) + i\theta^a(x)(T^a)_{ij}\psi_j(x)$
a ($SU(3)_c$)	$G^a_\mu(x) \to G^a_\mu(x) + f^{abc}\theta^b(x)G^c_\mu(x) + \frac{1}{g_3}\partial_\mu\theta^a(x)$
I ($SU(2)_L$)	$\psi_I(x) \to \psi_I(x) + i\theta^A(x)(\tau^A)_{IJ}\psi_J(x)$
A ($SU(2)_L$)	$W^A_\mu(x) \to W^A_\mu(x) + \epsilon^{ABC}\theta^B(x)W^C_\mu(x) + \frac{1}{g_2}\partial_\mu\theta^A(x)$
No index ($U(1)_Y$)	$\psi(x) \to \psi(x) + i\theta(x)Y\psi(x)$
α (Lorentz LH)	$\psi^\alpha(x) \to \psi^\alpha(x) - \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})^\alpha{}_\beta\psi^\beta(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\psi^\alpha(x)$
$\dot{\alpha}$ (Lorentz RH)	$\psi^{\dot{\alpha}}(x) \to \psi^{\dot{\alpha}}(x) + \frac{i}{2}\omega_{\mu\nu}((\sigma^{\mu\nu})^*)^{\dot{\alpha}}{}_{\dot{\beta}}\psi^{\dot{\beta}}(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\psi^{\dot{\alpha}}(x)$

Table 4: Infinitesimal Local Transformation Rules for Gauge and Spinor Indices

At the electroweak scale, the $SU(2)_L \times U(1)_Y$ gauge symmetry is spontaneously broken down to the electromagnetic subgroup $U(1)_{EM}$ by the Higgs mechanism. Consequently, the neutral component of the weak isospin gauge field (W_μ^3) and the hypercharge gauge field (B_μ) mix and yield the massless photon (A_μ) mediating Quantum Electrodynamics, and the massive Z^0 boson. Simultaneously, the orthogonal isospin components (W_μ^1, W_μ^2) reorganize into the massive $W^\pm$ bosons mediating the weak interaction. A detailed derivation of the spontaneous symmetry breaking dynamics and the Higgs sector falls beyond the goal of this study.

4 Faddeev–Popov Ghost

In perturbative quantum field theory, Feynman rules are extracted term by term due to the additive structure of the action ($S = S_{Matter} + S_{YM}$). The matter sector simply reproduces the standard QED-like fermion-gauge coupling ($\bar{\psi}\gamma^\mu A_\mu \psi$).

$$
\begin{aligned}
Z[I,J] &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A e^{iS(\psi,\bar{\psi},A,I,J)} && (20)\\
&= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A e^{i\{S_{Matter}(\psi,\bar{\psi},A,I)+S_{YM}(A,J)\}} && (21)
\end{aligned}
$$

where,

$$
\begin{aligned}
S_{Matter}(\psi,\bar{\psi},A,I) &= \int d^4x \left[\bar{\psi}(i\not{D} - m)\psi + I^\dagger \psi + \psi^\dagger I\right] && (22)\\
S_{YM}(A,J) &= \int d^4x \left[-\frac{1}{4}(F_{\mu\nu}^a)^2 + J^{a\mu} A_\mu^a\right] && (23)
\end{aligned}
$$

Unfortunately, this integral overcounts all equivalence classes of A under the gauge transformation defined in equation (18). To bypass this divergence, we introduce the gauge-fixing function $G^a(x) = \partial^\mu A_\mu^a(x) - \omega^a(x)$ dependent on an arbitrary function $\omega^a(x)$. In an explicit form, this implies:

$$
G^a(x) = 0 \iff \theta^a(x) - f(A_\mu^a(x)) = 0. \tag{24}
$$

Therefore, the path integral is gauge-fixed by inserting the Dirac delta function $\prod_{x,a} \delta(\theta - f(A))$, which isolates the gauge parameter and reduces the integration domain from $\mathcal{D}A = \mathcal{D}A_{fixed}\mathcal{D}\theta$ to $\mathcal{D}A_{fixed}$.

5 1-Loop Beta Function for Non-Abelian Gauge Theory

5.1 Basic Setting

This is the final explicit form of the effective Lagrangian expressed in terms of ψ, A, and c, using integration by parts.

$$\begin{aligned}\mathcal{L}_{eff} &= \bar{\psi}_i(i\not{\partial} - m)\psi_i + g\bar{\psi}_i\gamma^\mu A^a_\mu T^a_{ij}\psi_j \\ &+ \frac{1}{2}A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} + \frac{1}{2\xi}A^{a\mu}\partial_\mu\partial_\nu A^a_\nu - gf^{abc}\partial^\mu A^{a\nu}A^b_\mu A^c_\nu \\ &- \frac{1}{4}g^2 f^{abc}f^{ade}A^{\mu b}A^{\nu c}A^d_\mu A^e_\nu + \bar{c}^a\partial^\mu\partial_\mu c^a + gf^{abc}A^c_\mu(\partial^\mu\bar{c}^a)c^b\end{aligned} \tag{25}$$

In order to renormalize, we need to change the Lagrangian of bare fields into rescaled fields using Z factors of order of $1+\mathcal{O}(g^2)$. Then, it is equivalent to including counterterms in the original Lagrangian. In detail, rescaling is unnecessary for the gauge fixing term because it can be absorbed in ξ.

$$\begin{aligned}\mathcal{L}_{ct} &= i(Z_1-1)\bar{\psi}_i\not{\partial}\psi_j - (Z_2-1)m\bar{\psi}_i\psi_i + (Z_3-1)g\bar{\psi}_i\gamma^\mu A^a_\mu T^a_{ij}\psi_j \\ &+ \frac{1}{2}(Z_4-1)A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} - (Z_5-1)gf^{abc}\partial^\mu A^{a\nu}A^b_\mu A^c_\nu \\ &- \frac{1}{4}(Z_6-1)g^2 f^{abc}f^{ade}A^{\mu b}A^{\nu c}A^d_\mu A^e_\nu + (Z_7-1)\bar{c}^a\partial^\mu\partial_\mu c^a \\ &+ (Z_8-1)gf^{abc}A^c_\mu(\partial^\mu\bar{c}^a)c^b\end{aligned} \tag{26}$$

In dimension $4-\epsilon$, the fields and constants are rescaled as below, where $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$.

$$\psi_0 = Z_1^{1/2}\psi\tilde{\mu}^{-\frac{\epsilon}{2}} \tag{27}$$

$$m_0 = Z_1^{-1}Z_2 m \tag{28}$$

$$A_0 = Z_4^{1/2}A\tilde{\mu}^{-\frac{\epsilon}{2}} \tag{29}$$

$$c_0 = Z_7^{1/2}c\tilde{\mu}^{-\frac{\epsilon}{2}} \tag{30}$$

There are a total of five fundamental scaling factors for the degrees of freedom (ψ, m, A, c, g), but eight Z factors in the Lagrangian. The remaining three restrictions are manifested as equivalent expressions for the scaling of the coupling constant g.

$$g_0 = Z_1^{-1}Z_3Z_4^{-1/2}g\tilde{\mu}^{\frac{\epsilon}{2}} = Z_4^{-3/2}Z_5 g\tilde{\mu}^{\frac{\epsilon}{2}} = Z_4^{-1}Z_6^{1/2}g\tilde{\mu}^{\frac{\epsilon}{2}} = Z_4^{-1/2}Z_7^{-1}Z_8 g\tilde{\mu}^{\frac{\epsilon}{2}} \tag{31}$$

We need a delicate procedure to derive the beta functions. First, we find the Feynman rules for each term in the Lagrangian. Second, we evaluate the loop corrections and extract the Z factors required to derive the beta function (in this case, Z_1, Z_3, and Z_4 are sufficient). Finally, we derive the beta function from the invariance of the bare coupling g_0 with respect to the energy scale μ.

5.2 Feynmann Rules

Originally, extracting the Dyson series or proceeding via the path integral formalism is required to strictly derive the Feynman rules from the Lagrangian. However, the following general operational patterns are efficient for direct extraction.

For Quadratic Terms $\iff$ Propagators

- Extract the purely quadratic (free) terms from the Lagrangian.
- Convert the terms into momentum space by replacing the space-time derivatives with momentum vectors (typically $\partial_\mu \to ik_\mu$).
- Arrange the expression to isolate the operator matrix O between the fields, such as $\frac{1}{2}\phi_A O^{AB}(k)\phi_B$ for bosons or $\bar{\psi}O(k)\psi$ for fermions.
- Compute the inverse of the operator matrix O^{-1} and multiply it by i to obtain the exact Feynman propagator: $\Delta(k) = iO^{-1}$.

For Higher-Order Terms $\iff$ Vertices

- Extract the higher-order (interaction) terms from the Lagrangian.
- Multiply the entire interaction term by i.
- Convert any derivatives acting on the fields into corresponding momentum vectors ($\partial_\mu \to ik_\mu$ assuming all momenta k are defined as flowing into the vertex).
- Remove the field operators from the expression, leaving only the coupling constants, tensors, and momentum factors.
- If the interaction involves identical fields, you must fully symmetrize the rule by summing over all possible permutations of the identical particles' momenta and indices.

By this systematic procedure, the Feynman rules are uniquely determined from the associated Lagrangian terms.

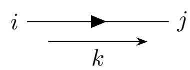
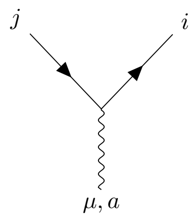
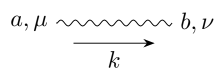
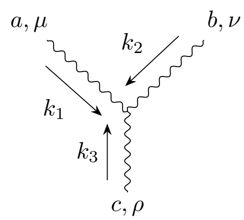
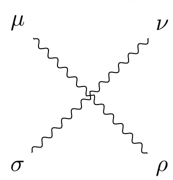
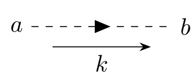
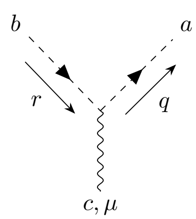
Diagram	**Lagrangian Term**	**Feynman Rule**
	$i\bar{\psi}_i \partial\!\!\!/ \psi_i - m\bar{\psi}_i\psi_i$	$\frac{i(k\!\!\!/ - m)}{k^2 + m^2 - i\epsilon}\delta_{ij}$
	$g\bar{\psi}_i\gamma^\mu A^a_\mu T^a_{ij}\psi_j$	$ig\gamma^\mu T^a_{ij}$
	$\frac{1}{2}A^{a\mu}\left[g_{\mu\nu}\partial^2 - (1-\frac{1}{\xi})\partial_\mu\partial_\nu\right]A^{a\nu}$	$\frac{-i\delta^{ab}}{k^2 - i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$
	$-gf^{abc}\partial^\mu A^{a\nu}A^b_\mu A^c_\nu$	$-\,gf^{abc}[g^{\mu\nu}(k_1-k_2)^\rho$ $+\,g^{\nu\rho}(k_2-k_3)^\mu$ $+\,g^{\rho\mu}(k_3-k_1)^\nu]$
	$-\frac{1}{4}g^2 f^{abc}f^{ade}A^{\mu b}A^{\nu c}A^d_\mu A^e_\nu$	$-\,ig^2\big[f^{eab}f^{ecd}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$ $+\,f^{eac}f^{edb}(g^{\mu\sigma}g^{\rho\nu} - g^{\mu\nu}g^{\rho\sigma})$ $+\,f^{eab}f^{ecd}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\sigma\nu})\big]$
	$\bar{c}^a\partial^\mu\partial_\mu c^a$	$\frac{-i\delta^{ab}}{k^2 - i\epsilon}$
	$gf^{abc}A^c_\mu(\partial^\mu\bar{c}^a)c^b$	$gf^{abc}q^\mu$

Table 5: Feynman Rules for Propagators and Vertices

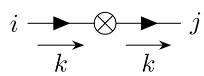
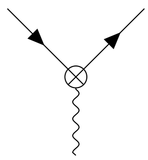
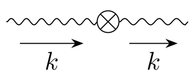
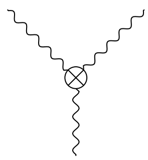
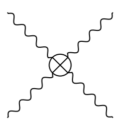
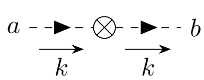
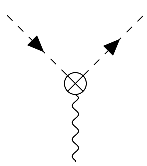
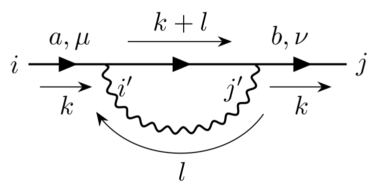
Diagram	**Lagrangian Term**	**Feynman Rule**
	$-i(Z_1-1)\bar{\psi}_i\not{\partial}\psi_i+(Z_2-1)m\bar{\psi}_i\psi_i$	$i\,[(Z_1-1)\not{k}+(Z_2-1)m]\,\delta_{ij}$
	$(Z_3-1)g\bar{\psi}_i\gamma^\mu A^a_\mu T^a_{ij}\psi_j$	$(Z_3-1)\times$ (Bare Rule)
	$\frac{1}{2}(Z_4-1)A^{a\mu}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A^{a\nu}$	$-i(Z_4-1)(k^2g^{\mu\nu}-k^\mu k^\nu)$
	$-(Z_5-1)gf^{abc}\partial^\mu A^{a\nu}A^b_\mu A^c_\nu$	$(Z_5-1)\times$ (Bare Rule)
	$-\frac{1}{4}(Z_6-1)g^2f^{abc}f^{ade}A^{\mu b}A^{\nu c}A^d_\mu A^e_\nu$	$(Z_6-1)\times$ (Bare Rule)
	$(Z_7-1)\partial^\mu\bar{c}^a\partial_\mu c^a$	$i(Z_7-1)\delta^{ab}k^2$
	$(Z_8-1)gf^{abc}A^c_\mu(\partial^\mu\bar{c}^a)c^b$	$(Z_8-1)\times$ (Bare Rule)

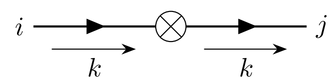
Table 6: Feynman Rules for Counterterms

5.3 1-Loop Correction

As previously mentioned, determining Z_1, Z_3, and Z_4 is sufficient to compute the beta function using equation (31). The components corresponding to these Z factors are the fermion propagator (specifically its $\not{k}$ dependence), the fermion-gauge boson vertex, and the gauge boson propagator, respectively. Because the loop diagram contains at least two vertices of order g, it is sufficient to ignore Z_g. Lastly, from the fact that the final results are irrelevant to arbitrary gauge fixing constant ξ, it is convenient to fix it to 1 for now.

For the fermion propagator, the relevant $\mathcal{O}(g^2)$ diagrams are shown below.





$$i\mathcal{M}_{11} \tag{32}$$

$$= i\int \frac{d^dl}{(2\pi)^d}[ig\tilde{\mu}^{\frac{\epsilon}{2}}\gamma^\nu T^b_{jj'}]\cdot[\frac{i(\not{k}+\not{l}-m)}{(k+l)^2+m^2-i\epsilon}\delta_{i'j'}]\cdot[ig\tilde{\mu}^{\frac{\epsilon}{2}}\gamma^\mu T^a_{ii'}]\cdot[\frac{-ig_{\mu\nu}}{l^2-i\epsilon}\delta^{ab}] \tag{33}$$

$$= -ig^2\tilde{\mu}^\epsilon C_2(r)\delta_{ij}\int\frac{d^dl}{(2\pi)^d}\frac{\gamma_\mu(\not{k}+\not{l}-m)\gamma^\mu}{\{(k+l)^2+m^2-i\epsilon\}\{l^2-i\epsilon\}} \tag{34}$$

$$= -ig^2\tilde{\mu}^\epsilon C_2(r)\delta_{ij}\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\frac{\gamma_\mu(\not{k}+\not{l}-m)\gamma^\mu}{(q^2+D)^2} \tag{35}$$

$$= -ig^2\tilde{\mu}^\epsilon C_2(r)\delta_{ij}\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\frac{-(d-2)(1-x)\not{k}-dm}{(q^2+D)^2} \tag{36}$$

$$= -ig^2\tilde{\mu}^\epsilon C_2(r)\delta_{ij}\int\frac{id^d\bar{q}}{(2\pi)^d}\int_0^1 dx\frac{-(2-\epsilon)(1-x)\not{k}-(4-\epsilon)m}{(\bar{q}^2+D)^2} \tag{37}$$

$$= g^2\tilde{\mu}^\epsilon C_2(r)\delta_{ij}\int_0^1 dx\frac{\Gamma(\frac{\epsilon}{2})\{-2(1-x)\not{k}-4m\}}{(4\pi)^{2-\frac{\epsilon}{2}}}D^{-\frac{\epsilon}{2}} \tag{38}$$

$$= \frac{g^2C_2(r)\delta_{ij}}{(4\pi)^2}\int_0^1 dx\{-2(1-x)\not{k}-4m\}\left(\frac{2}{\epsilon}-\gamma\right)\left(1-\frac{\epsilon}{2}\log\frac{D}{\mu^2}\right) \tag{39}$$

$$= \frac{g^2C_2(r)\delta_{ij}}{8\pi^2}\int_0^1 dx\{-2(1-x)\not{k}-4m\}\left(\frac{1}{\epsilon}-\frac{1}{2}\log\frac{D}{\mu^2}\right) \tag{40}$$

$$= -\left[\left\{\frac{g^2C_2(r)}{8\pi^2}\cdot\frac{1}{\epsilon}\right\}\not{k}+\left\{\frac{g^2C_2(G)}{2\pi^2}\cdot\frac{1}{\epsilon}\right\}m+\mathcal{O}(\epsilon^0)\right]\delta_{ij} \tag{41}$$

$$i\mathcal{M}_{12} = -\left[(Z_1-1)\not{k}+(Z_2-1)m\right]\delta_{ij} \tag{42}$$

where,

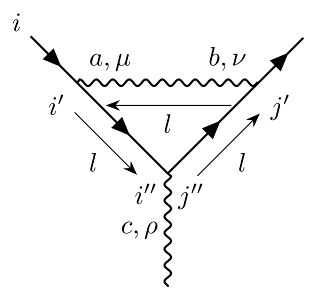
$$(T^a)^2_{ij} = C_2(r)\delta_{ij} \tag{43}$$

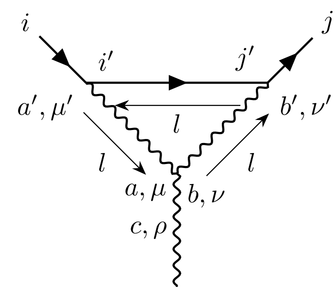
$$
\begin{aligned}
q &= l + kx && (44)\\
\bar{q}^0 &= iq^0 && (45)\\
\bar{q}^i &= q^i \quad (i \neq 0) && (46)\\
D &= x(1-x)k^2 + m^2 && (47)
\end{aligned}
$$

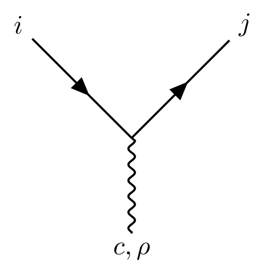
By selecting Z_1 to only cancel divergence ($\frac{1}{\epsilon}$) from $i(\mathcal{M}_{11} + \mathcal{M}_{12})$,

$$Z_1 = 1 - \frac{g^2 C_2(r)}{8\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(g^4). \tag{48}$$

For the fermion-gauge boson vertex, the relevant $\mathcal{O}(g^3)$ diagrams are shown below. Because we don't need to track specific momentum dependence term, it is sufficient to fix every external momentum 0.







$$
\begin{aligned}
& i\mathcal{M}_{31} && (49)\\
&= i \int \frac{d^d l}{(2\pi)^d} [ig\tilde{\mu}^{\frac{\epsilon}{2}} \gamma^\nu T^b_{jj'}] \cdot [\frac{i(\not{l} - m)}{l^2 + m^2 - i\epsilon} \delta_{j'j''}] \cdot && (50)\\
&[ig\tilde{\mu}^{\frac{\epsilon}{2}} \gamma^\rho T^c_{j''i''}] \cdot [\frac{i(\not{l} - m)}{l^2 + m^2 - i\epsilon} \delta_{i'i}] \cdot [ig\tilde{\mu}^{\frac{\epsilon}{2}} \gamma^\mu T^a_{ii'}] \cdot [\frac{-ig_{a\mu\nu}}{l^2 - i\epsilon} \delta^{ab}] && (51)\\
&= -ig^3 \tilde{\mu}^{\frac{3\epsilon}{2}} [C_2(r) - \frac{1}{2} C_2(G)] T^c_{ij} \int \frac{d^d l}{(2\pi)^d} \frac{\gamma_\mu (\not{l} - m) \gamma^\rho (\not{l} - m) \gamma^\mu}{\{l^2 + m^2 - i\epsilon\}\{l^2 + m^2 - i\epsilon\}\{l^2 - i\epsilon\}} && (52)
\end{aligned}
$$

$$= -ig^3\tilde{\mu}^{\frac{3\epsilon}{2}}[C_2(r) - \frac{1}{2}C_2(G)]T^c_{ij}\int \frac{d^dl}{(2\pi)^d}\int dF_3 \frac{\gamma_\mu(\not{l} - m)\gamma^\rho(\not{l} - m)\gamma^\mu}{(l^2 + D_1)^3} \tag{53}$$

$$= -ig^3\tilde{\mu}^{\frac{3\epsilon}{2}}[C_2(r) - \frac{1}{2}C_2(G)]T^c_{ij}\int \frac{id^d\bar{l}}{(2\pi)^d}\int dF_3 \frac{(d-2)^2}{d} \cdot \frac{\bar{l}^2\gamma^\rho + X^\rho}{(\bar{l}^2 + D_1)^3} \tag{54}$$

$$= -g^3\tilde{\mu}^{\frac{3\epsilon}{2}}[C_2(r) - \frac{1}{2}C_2(G)]T^c_{ij}\frac{(d-2)^2}{d}\int \frac{d^d\bar{l}}{(2\pi)^d}\int dF_3 \cdot \frac{\bar{l}^2\gamma^\rho + X^\rho}{(\bar{l}^2 + D_1)^3} \tag{55}$$

$$= -g^3\tilde{\mu}^{\frac{3\epsilon}{2}}[C_2(r) - \frac{1}{2}C_2(G)]T^c_{ij}\int dF_3 \frac{\Gamma(\frac{\epsilon}{2})\gamma^\rho D_1^{-\frac{\epsilon}{2}} + \frac{1}{2}X^\rho D_1^{-1}}{(4\pi)^{2-\frac{\epsilon}{2}}} \tag{56}$$

$$= -\frac{g^3\tilde{\mu}^{\frac{\epsilon}{2}}[C_2(r) - \frac{1}{2}C_2(G)]T^c_{ij}}{(4\pi)^2}\int dF_3 \left[\left(\frac{2}{\epsilon} - \gamma\right)\left(1 - \frac{\epsilon}{2}\log\frac{D_1}{\mu^2}\right)\gamma^\rho + \frac{X^\rho}{2D_1}\right] \tag{57}$$

$$= \left[-\frac{g^3\tilde{\mu}^{\frac{\epsilon}{2}}[C_2(r) - \frac{1}{2}C_2(G)]}{8\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right]\gamma^\rho T^c_{ij} \tag{58}$$

$$i\mathcal{M}_{32} \tag{59}$$

$$= i\int \frac{d^dl}{(2\pi)^d}[ig\tilde{\mu}^{\frac{\epsilon}{2}}\gamma^{\nu'}T^{b'}_{jj'}] \cdot [\frac{i(-\not{l} - m)}{l^2 + m^2 - i\epsilon}\delta_{i'j'}] \cdot [ig\gamma^{\mu'}T^{a'}_{i'i}] \cdot [\frac{-ig_{\mu\mu'}}{l^2 - i\epsilon}\delta^{aa'}]\cdot \tag{60}$$

$$[\frac{-ig\tilde{\mu}^{\frac{\epsilon}{2}}_{\nu\nu'}}{l^2 - i\epsilon}\delta^{bb'}] \cdot [-ig\tilde{\mu}^{\frac{\epsilon}{2}}f^{abc}\{g^{\mu\nu}(2l)^\rho - g^{\nu\rho}l^\mu - g^{\rho\mu}k^\nu\}] \tag{61}$$

$$= -ig^3\tilde{\mu}^{\frac{3\epsilon}{2}}C_2(G)T^c_{ij}\int \frac{d^dl}{(2\pi)^d}\frac{\gamma_\nu(\not{l} + m)\gamma_\mu\{g^{\mu\nu}(2l)^\rho - g^{\nu\rho}l^\mu - g^{\rho\mu}k^\nu\}}{(l^2 + m^2 - i\epsilon)(l^2 - i\epsilon)^2} \tag{62}$$

$$= -ig^3\tilde{\mu}^{\frac{3\epsilon}{2}}C_2(G)T^c_{ij}\int \frac{d^dl}{(2\pi)^d}\int dF_3 \frac{\gamma_\nu(\not{l} + m)\gamma_\mu\{g^{\mu\nu}(2l)^\rho - g^{\nu\rho}l^\mu - g^{\rho\mu}k^\nu\}}{(l^2 + D_2)^3} \tag{63}$$

$$= -ig^3\tilde{\mu}^{\frac{3\epsilon}{2}}C_2(G)T^c_{ij}\int \frac{id^d\bar{l}}{(2\pi)^d}\int dF_3 \frac{-\frac{4d-4}{d}\bar{l}^2}{(\bar{l}^2 + D_2)^3} \tag{64}$$

$$= -g^3\tilde{\mu}^{\frac{3\epsilon}{2}}C_2(G)T^c_{ij}\frac{4d-4}{d}\int \frac{d^d\bar{l}}{(2\pi)^d}\int dF_3 \frac{\bar{l}^2\gamma^\rho}{(\bar{l}^2 + D_2)^3} \tag{65}$$

$$= -3g^3\tilde{\mu}^{\frac{3\epsilon}{2}}C_2(G)T^c_{ij}\int dF_3 \frac{\Gamma(\frac{\epsilon}{2})\gamma^\rho}{(4\pi)^{2-\frac{\epsilon}{2}}}D_2^{-\frac{\epsilon}{2}} \tag{66}$$

$$= -\frac{3g^3\tilde{\mu}^{\frac{\epsilon}{2}}C_2(G)T^c_{ij}}{(4\pi)^2}\int dF_3 \left(\frac{2}{\epsilon} - \gamma\right)\left(1 - \frac{\epsilon}{2}\log\frac{D_3}{\mu^2}\gamma^\rho\right) \tag{67}$$

$$= \left[-\frac{3g^3\tilde{\mu}^{\frac{\epsilon}{2}}C_2(G)}{16\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right]\gamma^\rho T^c_{ij} \tag{68}$$

$$i\mathcal{M}_{33} = -(Z_3 - 1) g \tilde{\mu}^{\frac{\epsilon}{2}} \gamma^\rho T^c_{ij} \tag{69}$$

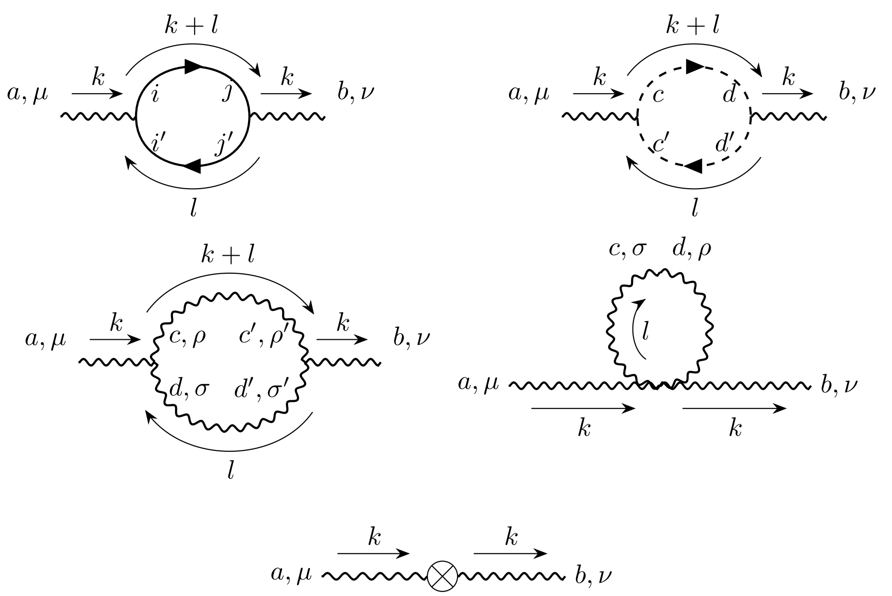
where,

$$f^{abc}(T^b T^a)_{ij} = C_2(G) T^c_{ij} \tag{70}$$
$$\bar{l}^0 = i l^0 \tag{71}$$
$$\bar{l}^i = l^i \quad (i \neq 0) \tag{72}$$
$$dF_3 = 2dxdydz\delta(x + y + z - 1) \tag{73}$$
$$D_1 = (x + y)m^2 \tag{74}$$
$$D_2 = xm^2 \tag{75}$$
$$X^\rho = \frac{d}{2-d}\gamma^\rho \tag{76}$$

By selecting Z_3 to only cancel divergence ($\frac{1}{\epsilon}$) from $i(\mathcal{M}_{31} + \mathcal{M}_{32} + \mathcal{M}_{33})$,

$$Z_3 = 1 - \frac{g^2[C_2(r) + C_2(G)]}{8\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(g^4). \tag{77}$$

Finally, for the gauge boson propagator, the relevant $\mathcal{O}(g^2)$ diagrams are shown below.



$$i\mathcal{M}_{41} \tag{78}$$

$$= (-1) \cdot n_f \cdot i \int \frac{d^d l}{(2\pi)^d} \mathrm{Tr}\Big[[(ig\tilde{\mu}^{\frac{\epsilon}{2}} \gamma^\mu T^a_{ij}] \cdot [\frac{i(\not{k} + \not{l} - m)}{(k+l)^2 + m^2 - i\epsilon} \delta_{ii'}] \cdot \tag{79}$$

$$[ig\tilde{\mu}^{\frac{\epsilon}{2}}\gamma^{\nu}T^{b}_{i'j'}]\cdot[\frac{i(\not{l}-m)}{l^2+m^2-i\epsilon}\delta_{jj'}]\Bigg] \tag{80}$$

$$= ig^2\tilde{\mu}^{\epsilon}n_fC(r)\delta^{ab}\int\frac{d^dl}{(2\pi)^d}\frac{\mathrm{Tr}[\gamma^{\mu}(\not{k}+\not{l}-m)\gamma^{\nu}(\not{l}-m)]}{\{(k+l)^2+m^2-i\epsilon\}\{l^2+m^2-i\epsilon\}} \tag{81}$$

$$= ig^2\tilde{\mu}^{\epsilon}n_fC(r)\delta^{ab}\int\frac{d^dq}{(2\pi)^d}\int_0^1 dx\frac{\left(\frac{2}{d}-1\right)(q^2g^{\mu\nu}+X_1^{\mu\nu})}{(q^2+D_1)^2} \tag{82}$$

$$= i\left(\frac{2}{d}-1\right)g^2\tilde{\mu}^{\epsilon}n_fC(r)\delta^{ab}\int\frac{id^d\bar{q}}{(2\pi)^d}\int_0^1 dx\frac{\bar{q}^2g^{\mu\nu}+X_1^{\mu\nu}}{(\bar{q}^2+D_1)^2} \tag{83}$$

$$= -\frac{g^2\tilde{\mu}^{\epsilon}n_fC(r)\delta^{ab}}{2}\int_0^1 dx\frac{\Gamma(-1+\frac{\epsilon}{2})g^{\mu\nu}D_1^{1-\frac{\epsilon}{2}}+\Gamma(-\frac{\epsilon}{2})X_1^{\mu\nu}D_1^{-\frac{\epsilon}{2}}}{(4\pi)^{2-\frac{\epsilon}{2}}} \tag{84}$$

$$= -\frac{g^2n_fC(r)\delta^{ab}}{32\pi^2}\int_0^1 dx\left\{-2D_1\left(\frac{2}{\epsilon}-\gamma+1\right)g^{\mu\nu}+\left(\frac{2}{\epsilon}-\gamma\right)X_1^{\mu\nu}\right\}\left(1-\frac{\epsilon}{2}\log\frac{D_1}{\mu^2}\right) \tag{85}$$

$$= \left[\frac{g^2n_fC(r)}{6\pi^2}\cdot\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right]k^2g^{\mu\nu}\delta^{ab}+\left[-\frac{g^2n_fC(r)}{6\pi^2}\cdot\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right]k^{\mu}k^{\nu}\delta^{ab} \tag{86}$$

$$i\mathcal{M}_{42} \tag{87}$$

$$= (-1)\cdot i\int\frac{d^dl}{(4\pi)^d}[g\tilde{\mu}^{\frac{\epsilon}{2}}f^{cda}(k+l)^{\mu}]\cdot[\frac{-i\delta^{cc'}}{(k+l)^2-i\epsilon}]\cdot[g\tilde{\mu}^{\frac{\epsilon}{2}}f^{d'c'b}l^{\nu}]\cdot[\frac{-i\delta^{dd'}}{l^2-i\epsilon}] \tag{88}$$

$$= ig^2\tilde{\mu}^{\epsilon}[-C_2(G)\delta^{ab}]\int\frac{d^dl}{(4\pi)^d}\frac{(k+l)^{\mu}l^{\nu}}{\{(k+l)^2-i\epsilon\}\{l^2-i\epsilon\}} \tag{89}$$

$$= -ig^2\tilde{\mu}^{\epsilon}C_2(G)\delta^{ab}\int\frac{d^dq}{(4\pi)^d}\int_0^1 dx\frac{X_2^{\mu\nu}}{(q^2+D_2)^2} \tag{90}$$

$$= -ig^2\tilde{\mu}^{\epsilon}C_2(G)\delta^{ab}\int\frac{id^d\bar{q}}{(4\pi)^d}\int_0^1 dx\frac{X_2^{\mu\nu}}{(\bar{q}^2+D_2)^2} \tag{91}$$

$$= g^2\tilde{\mu}^{\epsilon}C_2(G)\delta^{ab}\int_0^1 dx\Gamma(\frac{\epsilon}{2})X_2^{\mu\nu}D_2^{-\frac{\epsilon}{2}} \tag{92}$$

$$= g^2\tilde{\mu}^{\epsilon}C_2(G)\delta^{ab}\int_0^1 dx\frac{\Gamma(\frac{\epsilon}{2})X_2^{\mu\nu}D_2^{-\frac{\epsilon}{2}}}{(4\pi)^{2-\frac{\epsilon}{2}}} \tag{93}$$

$$= \frac{g^2C_2(G)\delta^{ab}}{(4\pi)^2}\int_0^1 dx\left(\frac{2}{\epsilon}-\gamma\right)X_2^{\mu\nu}\left(1-\frac{\epsilon}{2}\log\frac{D}{\mu^2}\right) \tag{94}$$

$$= \left[-\frac{g^2C_2(G)}{96\pi^2}\cdot\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right]k^2g^{\mu\nu}\delta^{ab}+\left[-\frac{g^2C_2(G)}{48\pi^2}\cdot\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right]k^{\mu}k^{\nu}\delta^{ab} \tag{95}$$

$$i\mathcal{M}_{43} \tag{96}$$

$$= \frac{1}{2} \cdot i \int \frac{d^d l}{(2\pi)^d} [-g\tilde{\mu}^{\frac{\epsilon}{2}} f^{acd} \{g^{\mu\rho}(2k+l)^\sigma + g^{\rho\sigma}(-k-2l)^\mu + g^{\sigma\mu}(-k+l)^\rho\}] \cdot [\frac{-ig_{\rho\rho'}}{(k+l)^2 - i\epsilon} \delta^{cc'}] \cdot \tag{97}$$

$$[-g\tilde{\mu}^{\frac{\epsilon}{2}} f^{bc'd'} \{g^{\nu\rho'}(-2k-l)^{\sigma'} + g^{\rho'\sigma'}(k+2l)^\mu + g^{\sigma'\mu}(k-l)^{\rho'}\}] \cdot [\frac{-ig_{\sigma\sigma'}}{(k+l)^2 - i\epsilon} \delta^{dd'}] \tag{98}$$

$$= -\frac{ig^2 \tilde{\mu}^\epsilon C_2(G) \delta^{ab}}{2} \int \frac{d^d l}{(2\pi)^d} \frac{\{g^{\mu\rho}(2k+l)^\sigma + g^{\rho\sigma}(-k-2l)^\mu + g^{\sigma\mu}(-k+l)^\rho\}}{\{k^2 - i\epsilon\}\{(k+l)^2 - i\epsilon\}} \cdot \tag{99}$$

$$\{\delta^\nu_\rho (-2k-l)^{\sigma'} + g_{\rho\sigma}(k+2l)^\mu + \delta^\sigma_\mu (k-l)^{\rho'}\} \tag{100}$$

$$= -\frac{ig^2 \tilde{\mu}^\epsilon C_2(G) \delta^{ab}}{2} \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \frac{6\left(-1+\frac{1}{d}\right)(q^2 g^{\mu\nu} + X_3^{\mu\nu})}{(q^2 + D_2)^2} \tag{101}$$

$$= -3i \frac{\left(-1+\frac{1}{d}\right) g^2 \tilde{\mu}^\epsilon C_2(G) \delta^{ab} \int \frac{id^d \bar{q}}{(2\pi)^d} \int_0^1 dx \bar{q}^2 g^{\mu\nu} + X_3^{\mu\nu}}{(\bar{q}^2 + D_2)^2} \tag{102}$$

$$= -\frac{9}{4} g^2 \tilde{\mu}^\epsilon C_2(G) \delta^{ab} \int_0^1 dx \frac{\Gamma(-1+\frac{\epsilon}{2}) g^{\mu\nu} D_2^{1-\frac{\epsilon}{2}} + \Gamma(-\frac{\epsilon}{2}) X_3^{\mu\nu} D_2^{-\frac{\epsilon}{2}}}{(4\pi)^{2-\frac{\epsilon}{2}}} \tag{103}$$

$$= -\frac{9}{4} g^2 C_2(G) \delta^{ab} \int_0^1 dx \left\{-2D_2 \left(\frac{2}{\epsilon} - \gamma + 1\right) g^{\mu\nu} + \left(\frac{2}{\epsilon} - \gamma\right) X_3^{\mu\nu}\right\} \left(1 - \frac{\epsilon}{2} \log \frac{D_2}{\mu^2}\right) \tag{104}$$

$$= -\frac{9g^2 C_2(G) \delta^{ab}}{64\pi^2} \int_0^1 dx \left\{-2D_2 \left(\frac{2}{\epsilon} - \gamma + 1\right) g^{\mu\nu} + \left(\frac{2}{\epsilon} - \gamma\right) X_3^{\mu\nu}\right\} \left(1 - \frac{\epsilon}{2} \log \frac{D_2}{\mu^2}\right) \tag{105}$$

$$= \left[-\frac{19g^2 C_2(G)}{96\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right] k^2 g^{\mu\nu} \delta^{ab} + \left[\frac{11g^2 C_2(G)}{48\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right] k^\mu k^\nu \delta^{ab} \tag{106}$$

$$i\mathcal{M}_{44} \sim \int d^d l \frac{1}{l^2 + m_\gamma^2} \sim \int d^d \bar{q} \frac{1}{\bar{q}^2 + m_\gamma^2} \sim m_\gamma^2 \cdot \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) = \mathcal{O}(\epsilon^0) \tag{107}$$

$$i\mathcal{M}_{45} = (Z_4 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \tag{108}$$

where n_f is the number of desperate ψ_i (analogous to the flavors of the quarks) and,

$$\mathrm{Tr}[T^a T^b] = C(r) \delta^{ab} \tag{109}$$

$$D_1 = x(1-x)k^2 + m^2 \tag{110}$$

$$D_2 = x(1-x)k^2 \tag{111}$$

$$X_1^{\mu\nu} = \frac{4d}{2-d} \left[\{x(1-x) - m^2\} k^2 g^{\mu\nu} - 2x(1-x) k^\mu k^\nu\right] \tag{112}$$

$$X_2^{\mu\nu} = -\frac{1}{2} x(1-x) k^2 g^{\mu\nu} - x(1-x) k^\mu k^\nu \tag{113}$$

$$X_3^{\mu\nu} = \frac{d}{6(-d+1)} \left[(5-2x+2x^2)k^2 g^{\mu\nu} - \{(d-6)+(-4d+8)x+(4d-6)x^2\}k^\mu k^\nu \right] \tag{114}$$

By selecting Z_4 to only cancel divergence $(\frac{1}{\epsilon})$ from $i(\mathcal{M}_{41}+\mathcal{M}_{42}+\mathcal{M}_{43}+\mathcal{M}_{44}+\mathcal{M}_{45})$,

$$Z_4 = 1 - \frac{g^2[4n_f C(r) - 5C_2(G)]}{24\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(g^4). \tag{115}$$

5.4 Derivation of Beta Function

Having determined the 1-loop corrections for Z_1, Z_3, and Z_4, we can now derive the beta function by enforcing the scale invariance of the bare parameters.

$$Z_1 = 1 - \frac{g^2 C_2(r)}{8\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(g^4) \tag{116}$$

$$Z_3 = 1 - \frac{g^2[C_2(r)+C_2(G)]}{8\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(g^4) \tag{117}$$

$$Z_4 = 1 - \frac{g^2[4n_f C(r) - 5C_2(G)]}{24\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(g^4) \tag{118}$$

Using the definitions $\alpha = \frac{g^2}{4\pi}$ and $Z_\alpha = Z_1^{-2} Z_3^2 Z_4^{-1}$, the bare coupling α_0 is given by:

$$\alpha_0 = Z_\alpha \alpha \tilde{\mu}^\epsilon \tag{119}$$

$$0 = \frac{\partial \log \alpha_0}{\partial \log \mu} = \frac{\partial \log Z_\alpha}{\partial \alpha} \frac{d\alpha}{d\log\mu} + \frac{1}{\alpha}\frac{d\alpha}{d\log\mu} + \epsilon \tag{120}$$

In the $\overline{\mathrm{MS}}$ scheme, $\log Z_\alpha$ is entirely defined by its pole structure and contains no positive powers of ϵ. Conversely, because the beta function $\frac{d\alpha}{d\log\mu}$ represents a physical, measurable observable, it must remain finite in the $\epsilon \to 0$ limit and therefore cannot contain singular $1/\epsilon$ poles. Summing up, their general Laurent expansions with respect to ϵ are strictly constrained:

$$\log Z_\alpha = \sum_{m=0}^{\infty} z_m(\alpha)\epsilon^{-m} \tag{121}$$

$$\frac{d\alpha}{d\log\mu} = \sum_{n=0}^{\infty} \beta_n(\alpha)\epsilon^n \tag{122}$$

Equation (120) must identically vanish for all values of α and ϵ. Requiring each coefficient of the Laurent series of ϵ to be zero immediately yields $\beta_n(\alpha) = 0$ for all $n \geq 2$. Matching the ϵ^0 and ϵ^1 terms produces the following system of equations:

$$\frac{dz_0}{d\alpha}\beta_0 + \frac{dz_1}{d\alpha}\beta_1 + \frac{1}{\alpha}\beta_0 = 0 \tag{123}$$

$$\frac{dz_0}{d\alpha}\beta_1 + \frac{1}{\alpha}\beta_1 + 1 = 0 \tag{124}$$

By expanding $\log Z_\alpha = -2\log Z_1 + 2\log Z_3 - \log Z_4 = -2(Z_1 - 1) + 2(Z_3 - 1) - (Z_4 - 1) + \mathcal{O}((Z_1 - 1)^2) + +\mathcal{O}((Z_3 - 1)^2) + +\mathcal{O}((Z_4 - 1)^2)$ and substituting the 1-loop results we've earned, we can extract the coefficients z_0 and z_1 up to $\mathcal{O}(\alpha^2)$:

$$z_0 = 1 + \mathcal{O}(\alpha^2) \tag{125}$$

$$z_1 = \frac{\alpha}{2\pi}\left\{\frac{4}{3}n_f C(r) - \frac{11}{3}C_2(G)\right\} + \mathcal{O}(\alpha^2) \tag{126}$$

Solving this system evaluates the components β_0 and β_1. Taking the physical limit $\epsilon \to 0$ isolates the final 4-dimensional beta function:

$$\beta(\alpha) = \frac{d\alpha}{d\log\mu} = \beta_0 + \beta_1\epsilon \xrightarrow{\epsilon\to 0} \frac{\alpha^2}{2\pi}\left\{\frac{4}{3}n_f C(r) - \frac{11}{3}C_2(G)\right\} + \mathcal{O}(\alpha^3) \tag{127}$$

where the intermediate variables are determined as:

$$\beta_1 = -\frac{1}{\frac{1}{\alpha} + \frac{dz_0}{d\alpha}} = -\alpha + \mathcal{O}(\alpha^2) \tag{128}$$

$$\beta_0 = -\frac{\frac{dz_1}{d\alpha}\beta_1}{\frac{1}{\alpha} + \frac{dz_0}{d\alpha}} = \frac{\alpha^2}{2\pi}\left\{\frac{4}{3}n_f C(r) - \frac{11}{3}C_2(G)\right\} + \mathcal{O}(\alpha^3) \tag{129}$$

6 Asymptotic Freedom

Having derived the beta function for a non-abelian gauge theory without scalar fields from Equation (127), we can determine the renormalization group (RG) evolution for each interaction in the Standard Model. This requires only specifying the quadratic Casimir invariant $C_2(G)$, the index of the representation $C(r)$, and the number of fermion flavors n_f. Recalling their standard definitions:

$$f^{acd}f^{bcd} = C_2(G)\delta^{ab} \tag{130}$$

$$\mathrm{Tr}[T^aT^b] = C(r)\delta^{ab} \tag{131}$$

Here, $C_2(G)$ is the quadratic Casimir invariant of the adjoint representation of the gauge group G. For $G = \mathrm{SU}(N)$, $C_2(G) = N$. Furthermore, $C(r)$ is the index of the representation r for the fermions. Since the Standard Model fermions transform under the fundamental representation of their respective gauge groups, $C(r) = \frac{1}{2}$.

In the specific case of Quantum Chromodynamics (QCD), the gauge group is $G = \mathrm{SU}(3)$, yielding $C_2(G) = 3$. There is a total of six quark flavors in nature (up, down, charm, strange, top, bottom), meaning $n_f = 6$. We do not count anti-quarks separately because their degrees of freedom are already inherently encoded within the structure of a single Dirac spinor field. Substituting these values yields the 1-loop beta function for QCD:

$$\beta_s(\alpha_s) = \frac{\alpha_s^2}{2\pi}\left\{\frac{4}{3}\cdot 6\cdot\frac{1}{2} - \frac{11}{3}\cdot 3\right\} + \mathcal{O}(\alpha_s^3) = -\frac{7\alpha_s^2}{2\pi} + \mathcal{O}(\alpha_s^3) \tag{132}$$

By integrating the beta function, we obtain the final scale dependence of the coupling constants for the toy QED model and QCD:

$$\alpha_{\mathrm{em}}(\mu) = \frac{\alpha_{\mathrm{em}}(\mu_0)}{1 - \frac{2\alpha_{\mathrm{em}}(\mu_0)}{3\pi}\ln\left(\frac{\mu}{\mu_0}\right)} \tag{133}$$

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{7\alpha_s(\mu_0)}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)} \tag{134}$$

Comparing the running couplings of QED and QCD reveals a stark contrast. While α_{em} increases as the energy scale μ increases, α_s steadily decreases. Translated to distance scales, the RG evolution of QCD dictates that the interaction strength grows as color charges are pulled further apart. This profound and counter-intuitive phenomenon is known as Asymptotic Freedom.

In each framework, there exists a specific divergence scale Λ in order to make α converge. Whether this scale functions as an upper or lower bound is dictated by the presence of asymptotic freedom, which strictly restricts the energy domain where perturbation theory remains valid. Furthermore, the Planck mass (M_{Pl}), the threshold where quantum gravitational phenomena must be incorporated, imposes an absolute upper bound on the effective theory.

$$\Lambda_{\mathrm{QED}} = \mu_0 \exp\left(\frac{3\pi}{2\alpha_{em}(\mu_0)}\right) \tag{135}$$

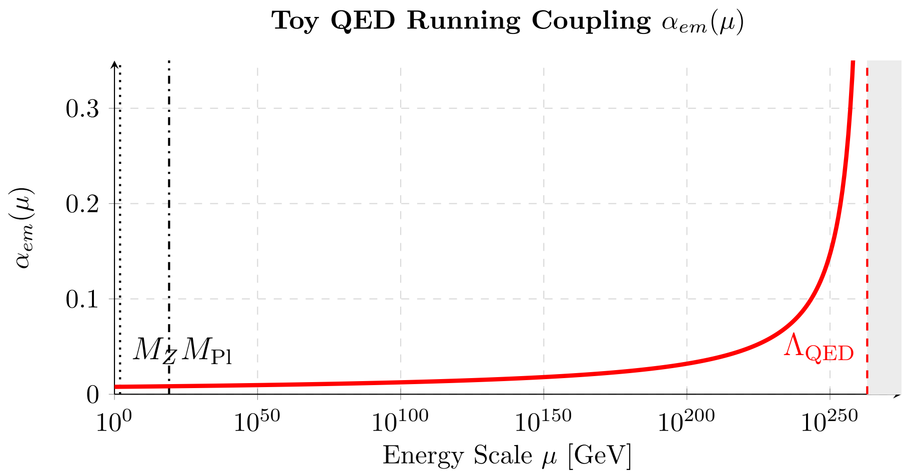
$$\Lambda_{\mathrm{QCD}} = \mu_0 \exp\left(-\frac{2\pi}{7\alpha_s(\mu_0)}\right) \tag{136}$$

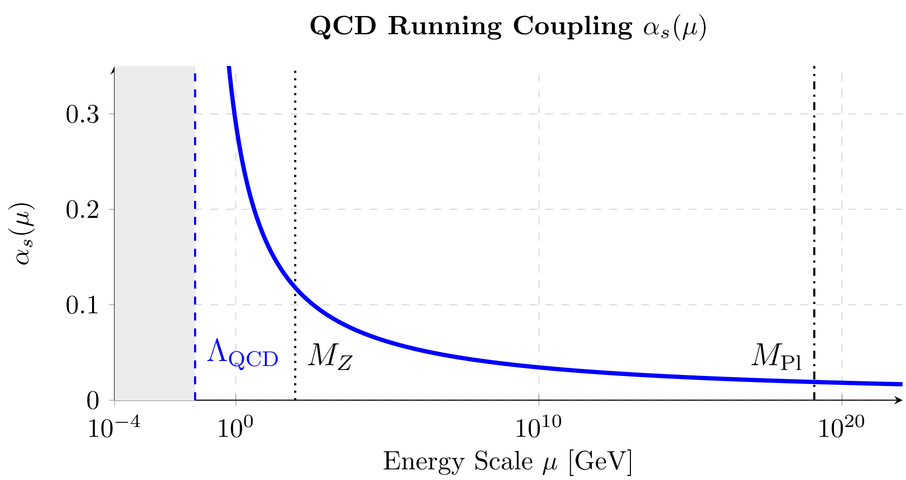
$$M_{\mathrm{Pl}} = \frac{1}{\sqrt{G}} \tag{137}$$

To perform direct calculations, initial reference parameters μ_0 and α_0 are required; herein, the Z boson mass (M_Z) is utilized as the baseline, as it is the most precisely measured and widely adopted standard from particle collider experiments.

Parameter / Scale	**Value**
Λ_{QED}	10^{263} GeV
Λ_{QCD}	0.0454 GeV
M_Z	91.1876 GeV
M_{Pl}	1.22×10^{19} GeV
$\alpha_{\text{em}}(M_Z)$	0.0078125
$\alpha_s(M_Z)$	0.118

Table 7: Numerical values of initial coupling constants and energy scales used in the RGE analysis





As a direct consequence of asymptotic freedom, interactions between color charges become highly unstable at large distances, inevitably resulting in color confinement. Consequently, the fundamental dynamics are governed by interactions at the intra-hadron scale, while interactions between composite hadrons are reduced to secondary residual effects, analogous to electric dipole interactions.

7 Further Study Goals

Further first objective is to incorporate the scalar Higgs field into the evaluation of the beta function, thereby enabling a complete RGE calculation for the remaining $U(1)_Y$ and $SU(2)_L$ gauge sectors to perform a comprehensive Grand Unified Theory (GUT) analysis.

Concurrently, we plan to investigate the Decoupling Theorem to rigorously understand the mechanics of how heavy particles decouple at low energies, effectively removing their dynamical contributions and driving the spontaneous splitting of the unified gauge group into separate subgroups.